New homogenization method for diffusion equations

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Abstract

In this paper, we propose and investigate a new homogenization method for diffusion problems in domains with multiple inclusions with large values of diffusion coefficients. The diffusion problem is approximated by the P1-finite element method on a triangular mesh. The underlying algebraic problem is replaced by a special system with a saddle point matrix. For the solution of the saddle point system we use the typical asymptotic expansion. We prove the error estimates and convergence of the expanded solutions. Numerical results confirm the theoretical conclusions.

Keywords: Diffusion equation, heterogeneous inclusion, homogenization.

MSC 2010: [
MSC 2010]76R50, 76M50, 35B27

Received: December 15, 2017

Homogenization method is important topic for theoretical research and applications [1, 2].

In this paper, we propose and investigate a new homogenization method for diffusion problems in domains with multiple inclusions with large values of diffusion coefficients. The diffusion problem is approximated by the P1-finite element method on a triangular mesh. The underlying algebraic problem is replaced by a special system with a saddle point matrix, as it was proposed in [5, 6, 3]. For the solution of the saddle point system we use the typical asymptotic expansion. We prove the error estimates and convergence of the expanded solutions. Numerical results confirm the theoretical conclusions.

The paper is organized as follows. In Section 1, we formulate the diffusion problem, and describe its approximation and the underlying matrices. We also describe the transformation of the classical finite element system with the symmetric positive definite matrix to an equivalent system with a saddle point matrix. In Section 2, we propose a new homogenization method for the solution of the saddle point problem and derive the error estimates. We formulate the condition for convergence of the expanded solutions to the solution of the saddle point system.

In Section 3, we derive an estimate for the parameter in the convergence condition for the expanded solution. Finally, in Section 4, we give numerical results for selected test problems relevant to practical applications. The numerical results clearly confirm the theoretical results in Section 2.

1 Problem formulation

In this paper, we consider the diffusion equation

$$-\nabla\left[k\left(x\right)\nabla u\right]=f,\quad x\in\Omega \tag{1.1}$$

in a polygonal domain $\Omega\in\mathbb{R}^2$ with the homogeneous Dirichlet boundary conditions on $\partial\Omega$, where $k\left(x\right)\geqslant 1$ is a positive piece-wise constant function.

Let ω_s be polygonal subdomains of Ω, $s=1,\ldots,m$, where m is a positive integer, such that $\overline{\omega}_s\cap\overline{\omega}_t=\varnothing$ and $\overline{\omega}_s\cap\partial\Omega=\varnothing,\ \ s,t=1,\ldots,m$. For the sake of simplicity we assume that subdomains ω_s are convex and

$$k\left(x\right)=\begin{cases}k_s=1+\dfrac{1}{\varepsilon_s}\equiv\text{const}>1 & \text{in }\omega_s,\ s=1,\ldots,m\\ 1, \quad \text{otherwise.}\end{cases} \tag{1.2}$$

We define the scalar product

$$\left(u,v\right)_0=\int\limits_{\Omega}\left(k\left(x\right)\nabla u\right)\cdot\nabla v,\quad u,v\in H_0^1\left(\Omega\right) \tag{1.3}$$

and denote by $(\,\cdot\,,\,\cdot\,)$ the scalar product in $L_2\left(\Omega\right)$.

Let Ω_h be a triangular mesh in Ω conforming with the boundary $\partial\omega_s$, i.e., $\partial\omega_s$ is the union of triangular sides in Ω_h, $s=1,\ldots,m$. We denote by V_h the classical P1-finite element subspace of $H_0^1\left(\Omega\right)$ on Ω_h. The P1-finite element method: Find $u_h\in V_h$, such that

$$\left(u_h,v_h\right)_0=\left(f,v_h\right)\quad\forall\, v_h\in V_h \tag{1.4}$$

results in the algebraic system

$$A_\varepsilon \overline{u} = \overline{f} \tag{1.5}$$

with the $N \times N$ matrix $A_\varepsilon = A_\varepsilon^T > 0$ and the vector $\overline{f} \in \mathbb{R}^N$, where N is the number of interior mesh nodes in Ω_h. With an appropriate ordering the matrix A_ε can be presented as 2×2 block matrix:

$$A_\varepsilon = \begin{bmatrix} A_{11,\varepsilon} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1.6}$$

with $n \times n$ submatrix

$$A_{11,\varepsilon} = A_{11} + B_{1,\varepsilon} \tag{1.7}$$

where $B_{1,\varepsilon}$ is the $m \times m$ block diagonal matrix with the diagonal blocks $\frac{1}{\varepsilon_s} A_s \in \mathbb{R}^{n_s \times n_s}$, $s = 1, \ldots, m$. Here, n_s is the number of nodes belonging to $\overline{\omega}_s$, $s = 1, \ldots, m$, $n = \sum_{s=1}^m n_s$, and matrices A_s are defined by the identities

$$(A_s \overline{u}_s, \overline{v}_s) = \int\limits_{\omega_s} \nabla u_h \cdot \nabla v_h \underset{\cdot}{\mathrm{x}} \quad \forall \, \overline{u}_s, \overline{v}_s \in \mathbb{R}^{n_s} \tag{1.8}$$

where $u_h, v_h \in V_{s,h}$ and $V_{s,h}$ is the restriction of V_h onto $\overline{\omega}_s$, $s = 1, \ldots, m$. In other words,

$$A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_{1,\varepsilon} & 0 \\ 0 & 0 \end{bmatrix} \tag{1.9}$$

where

$$B_{1,\varepsilon} = \operatorname{diag} \left\{ \frac{1}{\varepsilon_1} A_1, \ldots, \frac{1}{\varepsilon_m} A_m \right\}. \tag{1.10}$$

Using the procedure proposed in [5], we replace (1.5) by an equivalent system

$$\begin{array}{rcrcl} A\overline{u} & + & B^T \overline{p} & = & \overline{f} \\ B\overline{u} & - & \Sigma_\varepsilon B_1 \overline{p} & = & 0 \end{array} \tag{1.11}$$

where

$$B = [B_1 \; 0] \tag{1.12}$$

is $n \times N$ matrix,

$$B_1 = \operatorname{diag} \{A_1, \ldots, A_m\} \tag{1.13}$$

$$\Sigma_\varepsilon = \operatorname{diag} \{\varepsilon_1 I_1, \ldots, \varepsilon_m I_m\} \tag{1.14}$$

and

$$\overline{p} = \left[\Sigma_\varepsilon^{-1} \; 0\right] \overline{u}. \tag{1.15}$$

Here, I_s are the identity $n_s \times n_s$ matrices, $s = 1, \ldots, m$.

It is obvious, that the matrix

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & -\Sigma_\varepsilon B_1 \end{pmatrix} \tag{1.16}$$

is singular, $\dim(\ker \mathcal{A}) = m$, $\ker\, B^T = \ker\, B_1$, and any vector $\overline{w} \in \ker\, B_1$ can be presented in the block form by

$$\overline{w} = \begin{bmatrix} \overline{w}_1 \\ \vdots \\ \overline{w}_m \end{bmatrix} \tag{1.17}$$

where $\overline{w}_s \in \ker\, A_s$ are vectors with constant components, $s = 1, \ldots, m$. The solution vector $\overline{u} \in \mathbb{R}^N$ in (1.11) is unique, and the solution vector $\overline{p} \in \mathbb{R}^n$ in (1.11) is unique up to an arbitrary additive vector $\overline{w} \in \ker\, B_1$. In the future, we shall always assume that the solution vector $\overline{p}$ in (1.11) is orthogonal to $\ker\, B_1$, i.e., the uniqueness of $\overline{p}$.

2 Homogenization method

First, we consider system (1.11) with $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_m = 0$, i.e., the system

$$\begin{array}{rcrcl} A\overline{u}^0 & + & B^T\overline{p}^0 & = & \overline{f} \\ B\overline{u}^0 & & & = & 0 \end{array} \tag{2.1}$$

assuming that $\overline{p}^0 \perp \ker B_1$, and estimate the errors $\overline{u} - \overline{u}^0$ and $\overline{p} - \overline{p}^0$ in A_1-norm and B_1-seminorm, respectively. The error vectors satisfy the system

$$\begin{pmatrix} A & B^T \\ B & -\Sigma_\varepsilon B_1 \end{pmatrix} \begin{pmatrix} \overline{u} - \overline{u}^0 \\ \overline{p} - \overline{p}^0 \end{pmatrix} = \begin{pmatrix} 0 \\ \Sigma_\varepsilon B_1 \overline{p}^0 \end{pmatrix}. \tag{2.2}$$

Eliminating the vector $\overline{u} - \overline{u}^0$ in (2.2) we get the system

$$\left(S + \Sigma_\varepsilon B_1\right)\left(\overline{p} - \overline{p}^0\right) = -\Sigma_\varepsilon B_1 \overline{p}^0 \tag{2.3}$$

where

$$S = BA^{-1}B^T \equiv B_1 S_{11}^{-1} B_1 \tag{2.4}$$

and

$$S_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}. \tag{2.5}$$

Let us consider the eigenvalue problem

$$B_1 S_{11}^{-1} B_1 \overline{w} = \mu B_1 \overline{w} \tag{2.6}$$

with the eigenvalues $0 = \mu_1 = \mu_2 = \ldots = \mu_m < \mu_{m+1} \leqslant \ldots \leqslant \mu_n \leqslant 1$. The inequality $\mu_n \leqslant 1$ was proved in [3]. Then from (2.3) we derive the estimate

$$\left\| \overline{p} - \overline{p}^0 \right\|_{B_1} < \frac{\varepsilon}{\mu_{m+1}} \cdot \left\| \overline{p}^0 \right\|_{B_1} \tag{2.7}$$

where $\|\cdot\|_{B_1}$ denotes the semi-norm generated by the matrix B_1 and $\varepsilon = \max_{1,\ldots,m} \varepsilon_s$.

To estimate $\left\| \overline{p}^0 \right\|_{B_1}$ in (2.7) we return to system (2.1). Eliminating the vector $\overline{u}^0$ we obtain the equation

$$S\overline{p}^0 = -B_1 A^{-1} \overline{f}. \tag{2.8}$$

Then the estimate

$$\left\| \overline{p}^0 \right\|_{B_1} \leqslant \frac{1}{\mu_{m+1}} \left\| \overline{f} \right\|_{A^{-1}} \tag{2.9}$$

follows from the fact that

$$\left\| B_1^{1/2} A^{-1/2} \right\|_2 = \left\| A^{-1/2} B_1^{1/2} \right\|_2 \leqslant 1 \tag{2.10}$$

where $\|\cdot\|_{A^{-1}}$ is the norm generated by A^{-1}.

Thus, we get the estimate

$$\left\| \overline{p} - \overline{p}^0 \right\|_{B_1} < \frac{\varepsilon}{\mu_{m+1}^2} \cdot \left\| \overline{f} \right\|_{A^{-1}}. \tag{2.11}$$

Multiplying the first equation in (2.2) by the vector $\overline{u} - \overline{u}^0$, and applying the Schwartz inequality and (2.10), we get the estimates

$$\left\| \overline{u} - \overline{u}^0 \right\|_A \leqslant \left\| \overline{p} - \overline{p}^0 \right\|_{B_1} < \frac{\varepsilon}{\mu_{m+1}^2} \cdot \left\| \overline{f} \right\|_{A^{-1}}. \tag{2.12}$$

Now we assume that $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_m = \varepsilon$ and consider the expansions

$$\overline{u} = \sum_{s=0}^{\infty} \varepsilon^s \overline{u}^{(s)} \tag{2.13}$$

$$\overline{p} = \sum_{s=0}^{\infty} \varepsilon^s \overline{p}^{(s)} \tag{2.14}$$

where the vectors $\overline{u}^{(s)}$ and $\overline{p}^{(s)}$ satisfy equations (2.1) for $s = 0$ and the equations

$$\begin{array}{rcrcl} A\overline{u}^{(s)} & + & B^T\overline{p}^{(s)} & = & 0 \\ B\overline{u}^{(s)} & & & = & B_1\overline{p}^{(s-1)} \end{array} \tag{2.15}$$

for $s = 1, 2, \ldots$.

Let us define the error vectors

$$\overline{y}^{(s)} = \overline{u} - \sum_{l=0}^{s} \varepsilon^l \overline{u}^{(l)} \tag{2.16}$$

$$\overline{z}^{(s)} = \overline{p} - \sum_{l=0}^{s} \varepsilon^l \overline{p}^{(l)}. \tag{2.17}$$

It is obvious that the vectors $\overline{y}^{(s)}$ and $\overline{z}^{(s)}$ satisfy the system

$$\begin{array}{ccccc} A\overline{y}^{(s)} & + & B^T\overline{z}^{(s)} & = & 0 \\ B\overline{y}^{(s)} & - & \varepsilon B_1\overline{z}^{(s)} & = & -\varepsilon^s B_1\overline{p}^{(s)}. \end{array} \tag{2.18}$$

Eliminating the vector $\overline{y}^{(s)}$ in (2.18) we obtain the system

$$(S + \varepsilon B_1)\, \overline{z}^{(s)} = -\varepsilon B_1 \overline{p}^{(s)}. \tag{2.19}$$

Using the estimate

$$\left\| \overline{z}^{(s)} \right\|_{B_1} < \frac{\varepsilon}{\mu_{m+1}} \cdot \left\| \overline{p}^{(s)} \right\|_{B_1} \tag{2.20}$$

and estimates (2.9) and

$$\left\| \overline{p}^{(i)} \right\|_{B_1} < \frac{\varepsilon}{\mu_{m+1}} \cdot \left\| \overline{p}^{(i-1)} \right\|_{B_1}, \quad i = 1, \ldots, s \tag{2.21}$$

we get the final estimate

$$\left\| \overline{z}^{(s)} \right\|_{B_1} < \frac{\varepsilon^s}{\mu_{m+1}^{s+1}} \cdot \left\| \overline{f} \right\|_{A^{-1}}, \quad i = 1, \ldots, s. \tag{2.22}$$

Finally, from the first equation in (2.18) we derive the estimate

$$\left\| \overline{y}^{(s)} \right\|_{A} \leqslant \left\| \overline{z}^{(s)} \right\|_{B_1}. \tag{2.23}$$

On the basis of estimates (2.22) and (2.23) we obtain the following result.

Statement 2.1. The estimate

$$\left\| \overline{y}^{(s)} \right\|_{A} < \frac{\varepsilon^s}{\mu_{m+1}^{s+1}} \cdot \left\| \overline{u}^* \right\|_{A} \tag{2.24}$$

holds for any $s \geqslant 1$, where $\overline{u}^* = A^{-1}\overline{f}$.

Here, $\overline{u}_h^*$ is the P1-finite element solution of the Poisson problem

$$\begin{aligned} -\triangle u^* &= f \quad \text{in } \Omega \\ u^* &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.25}$$

on the mesh Ω_h.

Statement 2.2. The condition

$$\frac{\varepsilon}{\mu_{m+1}} < 1 \tag{2.26}$$

is sufficient for convergence of $u_h^{(s)}$ to u_h as $s \longrightarrow +\infty$.

3 Estimation from below for μ_{m+1}

We replace (2.6) by an equivalent eigenvalue problem: Find $\mu \in \mathbb{R}$, $\overline{v} \in \mathbb{R}^N$, $\overline{w} \in \mathbb{R}^n$, such that

$$\begin{array}{lclcl} A\overline{v} & + & B^T\overline{w} & = & 0 \\ B\overline{v} & & & = & -\mu B_1\overline{w} \end{array} \tag{3.1}$$

where

$$\overline{v} = -A^{-1}B^T\overline{w} \equiv \begin{bmatrix} \overline{v}_1 \\ \overline{v}_2 \end{bmatrix} \tag{3.2}$$

and $\overline{v}_1 = -S_{11}^{-1}B_1\overline{w}$. It follows, that

$$A_{21}\overline{v}_1 + A_{22}\overline{v}_2 = 0 \tag{3.3}$$

i.e., $\overline{v}_{2,h} \in V_{\Omega\setminus\omega,h}$ is the h-harmonic extension of $\overline{v}_{1,h} \in V_{\omega,h}$.

Here $V_{\omega,h}$ and $V_{\Omega\setminus\omega,h}$ are the restrictions of V_h onto $\overline{\omega}$ and $\Omega \setminus \omega$, respectively.

The finite element formulation of (3.1) is as follows: Find $\mu \in \mathbb{R}$, $\overline{v}_h \in V_h$ and $\overline{w}_h \in V_{\omega,h}$, such that

$$\begin{array}{lclcl} \int\limits_{\Omega} \nabla v_h \cdot \nabla\varphi_h \underset{\cdot}{x} & + & \int\limits_{\omega} \nabla w_h \cdot \nabla\varphi_h \underset{\cdot}{x} & = & 0 \\ \int\limits_{\omega} \nabla v_h \cdot \nabla\psi_h \underset{\cdot}{x} & & & = & -\mu \int\limits_{\omega} \nabla w_h \cdot \nabla\psi_h \underset{\cdot}{x} \end{array} \tag{3.4}$$

for any $\varphi \in V_h$, $\psi_h \in V_{\omega,h}$. To consider only nonzero eigenvalues we impose additional conditions

$$\int\limits_{\omega_s} w_h \underset{\cdot}{x} = 0, \quad s = 1, \ldots, m. \tag{3.5}$$

It is clear that μ_{m+1} in (2.6) is the minimal eigenvalue in (3.4)–(3.5). If we choose $\varphi_h = v_h$ and $\psi_h = w_h$ we obtain the equality

$$\int\limits_\Omega |\nabla v_h|^2 \underset{\cdot}{x} = \mu \int\limits_\omega |\nabla w_h|^2 \underset{\cdot}{x}. \tag{3.6}$$

Taking $\varphi_h = v_h$ and using the inequality

$$\left| \int\limits_\omega \nabla w_h \cdot \nabla v_h \underset{\cdot}{x} \right| \leqslant \left[\int\limits_\omega |\nabla w_h|^2 \underset{\cdot}{x} \right]^{1/2} \cdot \left[\int\limits_\Omega |\nabla v_h|^2 \underset{\cdot}{x} \right]^{1/2} \tag{3.7}$$

we obtain

$$\int\limits_\Omega |\nabla v_h|^2 \underset{\cdot}{x} \leqslant \int\limits_\omega |\nabla w_h|^2 \underset{\cdot}{x} \tag{3.8}$$

and conclude that the eigenvalues μ in (3.4)–(3.5) belong to the segment $(0;1]$.

Now we choose $\varphi_h \in V_h$ such that $\varphi_h = w_h$ in $\overline{\omega}$ and $\varphi_h = w_{2,h}$ in $\Omega \setminus \overline{\omega}$, where $w_{2,h}$ is a finite element extension of w_h from $\overline{\omega}$ into $\Omega \setminus \overline{\omega}$. Then from the first equation in (3.4) we obtain the equality

$$\int\limits_\Omega \nabla v_h \cdot \nabla \varphi_h \underset{\cdot}{x} = -\int\limits_\omega |\nabla w_h|^2 \underset{\cdot}{x} \tag{3.9}$$

which results in the inequality

$$\int\limits_\omega |\nabla w_h|^2 \underset{\cdot}{x} \leqslant \left[1 + C^2\right] \int\limits_\Omega |\nabla v_h|^2 \underset{\cdot}{x} \tag{3.10}$$

with

$$C^2 = \frac{\displaystyle\int\limits_{\Omega\setminus\overline{\omega}} |\nabla w_{2,h}|^2 \underset{\cdot}{x}}{\displaystyle\int\limits_\omega |\nabla w_h|^2 \underset{\cdot}{x}}. \tag{3.11}$$

To construct an appropriate function $w_{2,h}$ we define in Ω a set of open polygons $\hat{\omega}_s$, $s = 1, \ldots, m$, such that $\overline{\omega}_s \in \hat{\omega}_s$, i.e., $\partial\omega_s \cap \partial\hat{\omega}_s = \varphi$, $s = 1, \ldots, m$. An example of ω_s and $\hat{\omega}_s$ is given in Fig. 1.

We subject $w_{2,h}$ to the condition

$$w_{2,h} = 0 \quad \text{in } \Omega \setminus \hat{\omega} \tag{3.12}$$

where $\hat{\omega} = \bigcup_{s=1}^m \hat{\omega}_s$. Then,

$$C^2 = \max_{s=1,\ldots,m} C_s^2. \tag{3.13}$$

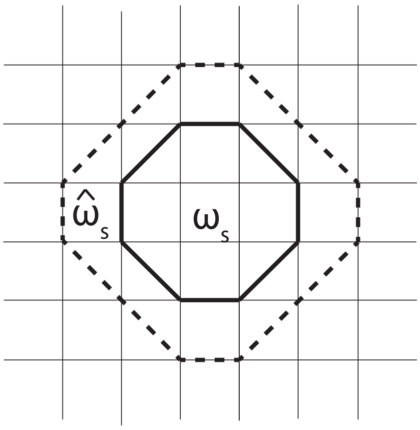

Figure 1: An example of ω_s and $\hat{\omega}_s$.

Here,

$$C_s^2 = \frac{\displaystyle\int\limits_{\hat{\omega}_s \setminus \omega_s} |\nabla w_h|^2 \underset{\cdot}{\mathrm{x}}}{\displaystyle\int\limits_{\omega_s} |\nabla w_h|^2 \underset{\cdot}{\mathrm{x}} + \beta_s^2 \left[\int\limits_{\omega_s} w_h \underset{\cdot}{\mathrm{x}} \right]^2} \tag{3.14}$$

due to above condition $\int_{\omega_s} w_h \underset{\cdot}{\mathrm{x}} = 0$, $s = 1, \dots, m$. We choose β_s^{-1} equal to area of ω_s, i.e., $\beta_s^{-1} = |\omega_s|$, $s = 1, \dots, m$.

We assume that the mesh Ω_h in $\hat{\omega}_s$ is conforming with respect to $\partial\hat{\omega}_s$, i.e., the boundaries of $\partial\hat{\omega}_s$ are unions of triangular mesh sides. We also assume that the traces of Ω_h on $\hat{\omega}_s$ are quasiuniform and regular shaped, $s = 1, \dots, m$. We observe that the norms $\| \cdot \|_S$ defined by

$$\|\xi\|_s^2 = \int\limits_{\hat{\omega}_s} |\nabla \xi|^2 \underset{\cdot}{\mathrm{x}} + \beta_s^2 \left[\int\limits_{\omega_s} \xi \underset{\cdot}{\mathrm{x}} \right]^2 \tag{3.15}$$

are equivalent to $H_0^1(\hat{\omega}_s)$-norms, $s = 1, \dots, m$. Then, it can be shown [7], that the expansion $w_{2,h}$ of w_h exists such that the values of C_s in (3.14) are independent of Ω_h. Such the extensions are said to be the norm preserving.

It was proved in [6] that with the additional assumptions

$$\operatorname{dist}(\partial\omega_s; \partial\hat{\omega}_s) = \min_{\substack{x \in \partial\omega_s \\ y \in \partial\hat{\omega}_s}} |x - y| \geqslant \alpha\, d_s \tag{3.16}$$

where α is a constant independent of $d_s = |\omega_s|^{1/2}$, the values of C_s in (3.14) are also independent of d_s, $s = 1, \dots, m$.

Thus, we have proved the following result.

ε	10^{-2}	10^{-4}	10^{-6}
$\delta u^{(0)}$	1.0e-2	1.0e-4	1.0e-6
$\delta p^{(0)}$	2.0e-2	2.0e-4	2.0e-6
$\delta u^{(1)}$	2.8e-4	2.9e-8	4.4e-10
$\delta p^{(1)}$	4.9e-4	5.1e-8	5.1e-12

Table 1: $h = d/2$.

ε	10^{-2}	10^{-4}	10^{-6}
$\delta u^{(0)}$	9.6e-3	9.9e-5	9.9e-7
$\delta p^{(0)}$	2.0e-2	2.0e-4	2.0e-6
$\delta u^{(1)}$	2.9e-4	3.0e-8	4.9e-10
$\delta p^{(1)}$	5.4e-4	5.5e-8	5.6e-12

Table 2: $h = d/4$.

ε	10^{-2}	10^{-4}	10^{-6}
$\delta u^{(0)}$	8.0e-3	8.3e-5	8.3e-7
$\delta p^{(0)}$	1.9e-2	2.0e-4	2.0e-6
$\delta u^{(1)}$	2.6e-4	2.7e-8	5.1e-10
$\delta p^{(1)}$	5.4e-4	5.6e-8	5.7e-12

Table 3: $h = d/8$.

$d = 2h$	$\varepsilon_{\max}$	10^{-2}	10^{-4}
	δu^0	4.2e-3	5.7e-5
	δp^0	6.5e-2	8.9e-4
$d = 4h$	$\varepsilon_{\max}$	10^{-2}	10^{-4}
	δu^0	4.3e-3	5.7e-5
	δp^0	6.0e-2	8.8e-4
$d = 8h$	$\varepsilon_{\max}$	10^{-2}	10^{-4}
	δu^0	4.1e-3	4.8e-5
	δp^0	6.6e-2	8.0e-4

Table 4: Random location of ω_s, $s = 1, \ldots, m$.

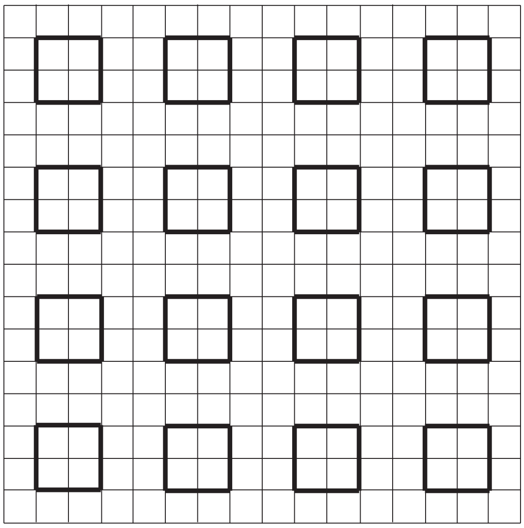

Figure 2: Periodic location of ω_s, $s = 1, \ldots, m$ with $d = 1/8$, $h = 1/16$, and $m = 16$.

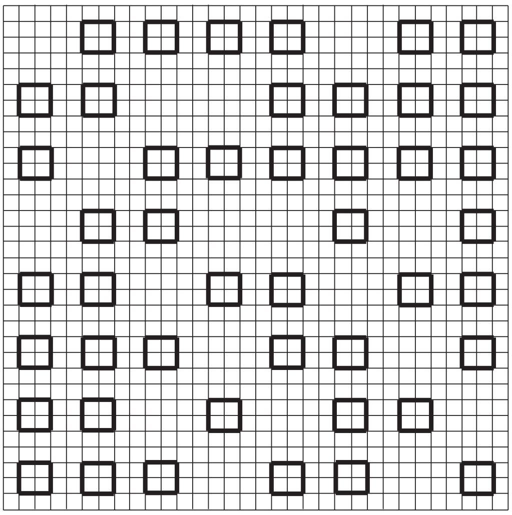

Figure 3: Random location of ω_s, $s = 1, \ldots, m$ with $d = 1/16$, $h = 1/32$, and $m = 46$.

Statement 3.1. Under all the above assumptions the estimate

$$\mu_{m+1} > \frac{1}{C^2 + 1} \tag{3.17}$$

holds with a constant C independent of Ω_h and the values of d_s, $s = 1, \ldots, m$.

Remark 3.1. Extension of the results in Sections 2 and 3 to 3D diffusion problems with inclusions on tetrahedral meshes, as well as to diffusion problems with different type of boundary conditions (Neumann, Robin, mixed) and to diffusion problems with nonzero reaction coefficients is straightforward.

4 Numerical results

Let Ω be the unite square and Ω_d be the square mesh in Ω with mesh step size $d = 1/\sqrt{m_d}$, where $\sqrt{m_d}$ is a positive integer. We define the set of meshes Ω_h in Ω with the mesh step sizes $h = d/2^l$, $l = 1, 2, \ldots$. We define the set of ω_s, $s = 1, \ldots, m$, where $m \leqslant m_d$, as the set of $d \times d$ squares with the centers in the nodes of the mesh Ω_d. Distribution of ω_s in Ω is random. Examples of meshes Ω_h with inclusions ω_s, $s = 1, \ldots, m$, are shown in Figs. 2 and 3.

In numerical tests we computed the values

$$\delta\overline{u}^{(s)} = \frac{\|\overline{u} - \overline{u}^{(s)}\|_A}{\|\overline{f}\|_{A^{-1}}}, \quad \delta\overline{p}^{(s)} = \frac{\|\overline{p} - \overline{p}^{(s)}\|_{B_1}}{\|\overline{f}\|_{A^{-1}}}, \quad s = 0, 1, 2. \tag{4.1}$$

In Tables 1–3 we show the results for $m = m_d$, i.e., for periodic location of inclusions, on the refined meshes with $h = d/2$, $h = d/4$, and $h = d/8$ for three constant values of ε equal to 10^{-2}, 10^{-4}, and 10^{-6}.

In Table 4, we show the results on random location of inclusions ω_s and randomly chosen values of $\varepsilon_s \in \left[10^{-6}, \varepsilon_{\max}\right]$, $s = 1, \ldots, m$. Table 4 shows the results for $d = 2h$, $d = 4h$, and $d = 8h$.

In all above tables, the numerical results clearly confirm the estimates derived in Section 2. More numerical results will be given in the forthcoming publication [4].

Acknowledgement: The author is grateful to V. K. Kramarenko for numerical results and assistance in the preparation of the paper.

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