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A TWO-ENERGIES PRINCIPLE FOR THE BIHARMONIC EQUATION AND AN A POSTERIORI ERROR ESTIMATOR FOR AN INTERIOR PENALTY DISCONTINUOUS GALERKIN APPROXIMATION

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Abstract. We consider an a posteriori error estimator for the Interior Penalty Discontinuous Galerkin (IPDG) approximation of the biharmonic equation based on the Hellan-Herrmann-Johnson (HHJ) mixed formulation. The error estimator is derived from a two-energies principle for the HHJ formulation and amounts to the construction of an equilibrated moment tensor which is done by local interpolation. The reliability estimate is a direct consequence of the two-energies principle and does not involve generic constants except for possible data oscillations. The efficiency of the estimator follows by showing that it can be bounded from above by a residual-type estimator known to be efficient. A documentation of numerical results illustrates the performance of the estimator.

Key words. biharmonic equation, two-energies principle, interior penalty discontinuous Galerkin method, a posteriori error estimator, equilibration

AMS subject classifications. 35J35, 65N30, 65N50

1. Introduction.

The biharmonic equation is more often solved by nonconforming or mixed methods than by conforming elements in order to avoid the computationally expensive implementation of H^2 conforming elements such as the Argyris plate elements of the TUBA family [4] or the generalizations of the Hsieh–Clough–Tocher elements from [20]. As far as mixed methods are concerned, the fourth order equation is written as a system of two second order equations, e.g.,

$$\begin{aligned} D^2u &= \underline{\underline{\mathbf{p}}}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{p}}} &= f, \end{aligned} \tag{1.1}$$

where D^2u is the matrix of second partial derivatives of u and $\underline{\underline{\mathbf{p}}}$ stands for the moment tensor. The formulation (1.1) leads to the mixed method of Hellan–Herrmann–Johnson [30, 31, 33]. Another splitting is given by

$$\begin{aligned} \Delta u &= w, \\ \Delta w &= f, \end{aligned} \tag{1.2}$$

and leads to the mixed method of Ciarlet–Raviart [17]. Among nonconforming approaches, Discontinuous Galerkin (DG) methods have been studied recently in [14, 15, 26, 27, 28] (for other fourth order problems see [21, 42]). The relationship between DG methods and mixed methods turns out to be useful for the biharmonic problem as it is for second order elliptic boundary value problems due to the unified analysis in [6]. Fourth order problems have been treated similarly in [27].

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The Interior Penalty DG (IPDG) methods considered in [27, 28] rely on the Ciarlet–Raviart mixed formulation (1.2). They are fully discontinuous in the sense that globally discontinuous, piecewise polynomials of degree $k \geq 2$ are used for the approximation of the primal variable u. On the other hand, those in [14, 15, 26] are based on the Hellan–Herrmann–Johnson splitting as given by (1.1). The IPDG schemes in [14, 15, 26] feature C^0 elements of Lagrangian type. Residual-type a posteriori error estimators have been considered and analyzed in [14, 26], and [28].

We will consider a posteriori error bounds by the two-energies principle, also known as the hypercircle method. It was originally developed by Prager and Synge [36, 38, 39] and more recently considered in connection with second order elliptic problems in [1, 7, 8, 9, 10, 11, 12, 41]. The considerations of DG methods in this direction [2, 3, 18, 22, 23, 24, 25] were also done for equations of second order.

In this paper, we focus on the biharmonic equation in the formulation of Hellan–Herrmann–Johnson and the application of the hypercircle method to its IPDG approximation. The advantage of a posteriori error bounds based on the two-energies principle compared to standard residual-type error estimators is that the reliability estimate does not contain generic constants except for possible oscillation terms (see the papers mentioned above and (5.8) below). As we shall see, the implementation amounts to the construction of an equilibrated moment tensor which can be done by means of a discrete three-field mixed formulation of the IPDG approximation. The construction only requires local interpolations in a postprocessing. Nevertheless, the analysis is more involved than the analogous one for equations of second order.

The paper is organized as follows: Section 2 lists some notation. In Section 3, we introduce the two-energies principle for the Hellan–Herrmann–Johnson mixed formulation (1.1). Section 4 is devoted to the IPDG approximation and associated discrete two-field and three-field formulations. Section 5 describes how the error bounds obtained from the two-energies principle can be built into a reliable a posteriori error estimator. The construction of the equilibrated moment tensor is dealt with in Section 6. In Section 7, we prove the efficiency of the estimator by showing that it can be bounded from above by a residual-type estimator which is known to be efficient. Finally, in Section 8 we provide a documentation of numerical results illustrating the quasi-optimality of the IPDG approximation and the performance of the estimator.

2. Notation. We will use standard notation from Lebesgue and Sobolev space theory [8, 13, 40]. In particular, for a bounded domain $\Omega \subset \mathbb{R}^2$ and $D \subseteq \bar{\Omega}$ we denote the L^2-inner product and the associated L^2-norm by $(\cdot,\cdot)_{0,D}$ and $\|\cdot\|_{0,D}$, respectively. We further refer to $H^k(\Omega), k \in \mathbb{N}$, as the Sobolev spaces with inner product $(\cdot,\cdot)_{k,\Omega}$, norm $\|\cdot\|_{k,\Omega}$, and seminorm $|\cdot|_{k,\Omega}$, and to $H^{k-1/2}(\Gamma'), \Gamma' \subseteq \Gamma = \partial\Omega$, as the associated trace spaces. $H_0^k(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in the H^k-norm. Further, $H^{-k}(\Omega)$ refers to the dual space of $H_0^k(\Omega)$ with $\langle\cdot,\cdot\rangle_{k,\Omega}$ denoting the dual product. Moreover, $\mathbf{H}(\mathrm{div},\Omega)$ is the Hilbert space of vector fields $\underline{\mathbf{q}} \in L^2(\Omega)^2$ such that $\nabla \cdot \underline{\mathbf{q}} \in L^2(\Omega)$. Matrix-valued functions in $L^2(\Omega)^{2\times 2}$ will be denoted by $\underline{\underline{\mathbf{q}}} = (q_{ij})_{i,j=1}^2$ and the inner-product is $(\underline{\underline{\mathbf{p}}},\underline{\underline{\mathbf{q}}})_{0,\Omega} := \int_\Omega \underline{\underline{\mathbf{p}}} : \underline{\underline{\mathbf{q}}}\, dx$, where $\underline{\underline{\mathbf{p}}} : \underline{\underline{\mathbf{q}}} := \sum_{i,j=1}^2 p_{ij}q_{ij}$. Further, we introduce the Hilbert space

$$\underline{\underline{\mathbf{H}}}(\mathrm{div}^2,\Omega) := \{\underline{\underline{\mathbf{q}}} \in \mathbf{H}(\mathrm{div},\Omega)^2 \mid \boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{q}}} \in \mathbf{H}(\mathrm{div},\Omega)\}.$$

Finally, given a function $u \in H^2(\Omega)$, we refer to $D^2u := (\partial^2 u/\partial x_i \partial x_j)_{i,j=1}^2$ as the matrix of second partial derivatives.

Let $\mathcal{T}_h(\Omega)$ be a geometrically conforming, locally quasi-uniform simplicial triangulation of the computational domain. For $D \subseteq \overline{\Omega}$, we denote by $\mathcal{E}_h(D)$ the set of edges of $\mathcal{T}_h(\Omega)$ in D. We further denote by $h_K, K \in \mathcal{T}_h(\Omega)$, the diameter of K and by $h_E, E \in \mathcal{E}_h(\overline{\Omega})$, the length of E. Moreover, for $D \subseteq K$ we refer to $P_m(D)$, $m \in \mathbb{N}$, as the set of polynomials of degree $\leq m$ on D. Due to the local quasi-uniformity of the triangulation, there exist constants $0 < c \leq C$ such that

$$c\, h_E \leq h_K \leq C\, h_E, \quad E \in \mathcal{E}_h(\partial K). \tag{2.1}$$

For a function $w \in L^2(\Omega)$ with $w|_K \in C(K), K \in \mathcal{T}_h(\Omega)$, and an interior edge $E = K_+ \cap K_-,\ K_\pm \in \mathcal{T}_h(\Omega)$, we set $w^\pm := w|_{E\cap K_\pm}$ and define the average and jump across E as usual according to

$$\{w\}_E := \begin{cases} \frac{1}{2}\,(w^+ + w^-)\ , & E \in \mathcal{E}_h(\Omega) \\ w|_E\ , & E \in \mathcal{E}_h(\Gamma) \end{cases}, \tag{2.2a}$$

$$[w]_E := \begin{cases} w^+ - w^-\ , & E \in \mathcal{E}_h(\Omega) \\ w|_E\ , & E \in \mathcal{E}_h(\Gamma) \end{cases}. \tag{2.2b}$$

The average and jump across $E \in \mathcal{E}_h(\bar{\Omega})$ are defined analogously for vector fields $\mathbf{w} \in L^2(\Omega)^2$ with $\mathbf{w}|_K \in C(K)^2, K \in \mathcal{T}_h(\Omega)$, and tensors $\underline{\underline{\mathbf{p}}} \in L^2(\Omega)^{2\times 2}$ with $\underline{\underline{\mathbf{p}}}|_K \in C(K)^{2\times 2}, K \in \mathcal{T}_h(\Omega)$. Moreover, we refer to $\mathbf{n}_E, E \in \mathcal{E}_h(\Omega), E = K_+ \cap K_-$, as the unit normal vector pointing from K_+ to K_- and to $\mathbf{n}_E, E \in \mathcal{E}_h(\Gamma)$, as the exterior unit normal vector $\mathbf{n}_\Gamma$ on $E \cap \Gamma$. Products like

$$[w]_E \mathbf{n}_E = w^+ \mathbf{n}_{\partial K_+} + w^- \mathbf{n}_{\partial K_-}$$

and other products under consideration are independent of the choice of K_+ and K_- and the resulting orientation of the edge.

3. A two-energies principle for the biharmonic equation. Given a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma := \partial\Omega$ and a function $f \in H^{-2}(\Omega)$, we consider the biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{3.1a}$$

$$u = \mathbf{n}_\Gamma \cdot \boldsymbol{\nabla} u = 0 \quad \text{on } \Gamma. \tag{3.1b}$$

A primal variational formulation of (3.1) amounts to the computation of $u \in H_0^2(\Omega)$ such that for all $v \in H_0^2(\Omega)$ it holds

$$(D^2 u, D^2 v)_{0,\Omega} = \langle f, v\rangle_{2,\Omega}. \tag{3.2}$$

It is well-known that (3.2) represents the optimality condition for the following unconstrained minimization problem: Find $u \in H_0^2(\Omega)$ such that

$$J_p(u) = \inf_{v\in H_0^2(\Omega)} J_p(v),$$

where the primal energy functional $J_p : H_0^2(\Omega) \to \mathbb{R}$ is given by

$$J_p(v) := \frac{1}{2}(D^2 v, D^2 v)_{0,\Omega} - \langle f, v\rangle_{2,\Omega}. \tag{3.3}$$

In order to specify the associated dual problem, the divergence of a matrix-valued function $\underline{\underline{\mathbf{q}}} = (q_{ij})_{i,j=1}^2$ with row vectors $\underline{\mathbf{q}}^{(i)} = (q_{i1}, q_{i2})^T, 1 \leq i \leq 2$, is defined as usual

$$\boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{q}}} := (\boldsymbol{\nabla} \cdot \underline{\mathbf{q}}^{(1)}, \boldsymbol{\nabla} \cdot \underline{\mathbf{q}}^{(2)})^T. \tag{3.4}$$

The dual or complementary energy $J_d : L^2(\Omega)^{2\times 2} \to \mathbb{R}$, given by

$$J_d(\underline{\underline{\mathbf{q}}}) := -\frac{1}{2}(\underline{\underline{\mathbf{q}}}, \underline{\underline{\mathbf{q}}})_{0,\Omega},$$

will be maximized subject to the constraint

$$(\underline{\underline{\mathbf{q}}}, D^2 v)_{0,\Omega} = \langle f, v\rangle_{2,\Omega} \quad \text{for all } v \in H_0^2(\Omega). \tag{3.5}$$

The relation (3.5) may be understood as

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{q}}} = f \quad \text{in } H^{-2}(\Omega)$$

or in the distributional sense.

Theorem 3.1. *Let J_p and J_d be defined as above. Then*

$$\min_{v\in H_0^2} J_p(v) = \max_{\underline{\underline{\mathbf{q}}}\in L^2(\Omega)^{2\times 2}} \left\{ J_d(\underline{\underline{\mathbf{q}}}) \mid \boldsymbol{\nabla} \cdot \nabla \cdot \underline{\underline{\mathbf{q}}} = f \right\} \tag{3.6}$$

where the constraint on the right-hand side of (3.6) *is understood as in* (3.5).

Proof. By definition we have for v and $\underline{\underline{\mathbf{q}}}$ as in (3.6)

$$\begin{aligned}
J_p(v) - J_d(\underline{\underline{\mathbf{q}}}) &= \frac{1}{2}(D^2 v, D^2 v)_{0,\Omega} - \langle f, v\rangle_{2,\Omega} + \frac{1}{2}(\underline{\underline{\mathbf{q}}}, \underline{\underline{\mathbf{q}}})_{0,\Omega} \\
&= \frac{1}{2}(D^2 v - \underline{\underline{\mathbf{q}}},\, D^2 v - \underline{\underline{\mathbf{q}}})_{0,\Omega} + (\underline{\underline{\mathbf{q}}}, D^2 v)_{0,\Omega} - (f, v)_{0,\Omega} \\
&= \frac{1}{2}\|D^2 v - \underline{\underline{\mathbf{q}}}\|_{0,\Omega} \geq 0,
\end{aligned}$$

since the relation (3.5) holds by assumption. It follows that $\inf J_p(v) \geq \sup J_d(\underline{\underline{\mathbf{q}}})$ where the infimum and the supremum are understood in the spirit of (3.6). Since we have equality for $v := u$ and $\underline{\underline{\mathbf{q}}} := D^2 u$, the proof is complete. ∎

We are now in a position to state an abstract version of the two-energies principle for the biharmonic equation; cf. [35, Theorem 3.1].

Theorem 3.2. *(Two-energies principle for the biharmonic equation)*
Let $u \in H_0^2(\Omega)$ be the solution of (3.2), *and let $\underline{\underline{\mathbf{p}}} \in L^2(\Omega)^{2\times 2}$ satisfy the equilibrium condition*

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{p}}} = f \quad \text{in } H^{-2}(\Omega). \tag{3.7}$$

Then, for $v \in H_0^2(\Omega)$ it holds

$$\|D^2 v - \underline{\underline{\mathbf{p}}}\|_{0,\Omega}^2 = \|D^2(v - u)\|_{0,\Omega}^2 + \|D^2 u - \underline{\underline{\mathbf{p}}}\|_{0,\Omega}^2. \tag{3.8}$$

Proof. We provide a short proof for completeness. If $u \in H_0^2(\Omega)$ is the solution of (3.2), then $(D^2u, D^2(v-u))_{0,\Omega} = \langle f, v-u \rangle_{2,\Omega}$. Next we conclude from (3.5) that the equilibrium assumption (3.7) implies $(\underline{\underline{\mathbf{p}}}, D^2(v-u))_{0,\Omega} = \langle f, v-u \rangle_{2,\Omega}$. Hence,

$$(D^2u - \underline{\underline{\mathbf{p}}}, D^2(u-v))_{0,\Omega} = \langle f - f, u - v \rangle_{2,\Omega} = 0.$$

An application of the binomial formula to $\|(D^2v - D^2u) + (D^2u - \underline{\underline{\mathbf{p}}})\|_{0,\Omega}^2$ yields (3.8). □

The relationship (3.8) is called the two-energies principle, because it can be stated in terms of the primal energy $J_p(v)$ and the complementary energy $J_d(\underline{\underline{\mathbf{p}}})$ as

$$\|D^2(v-u)\|_{0,\Omega}^2 + \|D^2u - \underline{\underline{\mathbf{p}}}\|_{0,\Omega}^2 = 2\big(J_p(v) - J_d(\underline{\underline{\mathbf{p}}})\big).$$

We conclude this section with a formulation of the two-energies principle that is better manageable in finite element computations. In particular, it translates the equilibrium condition (3.7) for $f \in L^2(\Omega)$ from H^{-2} to an element-wise property. We consider moment tensors $\underline{\underline{\mathbf{p}}} \in L^2(\Omega)^{2\times 2}$ that satisfy

$$\underline{\underline{\mathbf{p}}}|_K \in P_k(K)^{2\times 2}, \quad k \geq 2,\ K \in \mathcal{T}_h(\Omega), \tag{3.9a}$$

$$[\underline{\underline{\mathbf{p}}}]_E\ \mathbf{n}_E = 0, \qquad E \in \mathcal{E}_h(\Omega), \tag{3.9b}$$

$$\mathbf{n}_E \cdot [\boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{p}}}]_E = 0, \quad E \in \mathcal{E}_h(\Omega). \tag{3.9c}$$

The propertiest (3.9) imply $\underline{\underline{\mathbf{p}}} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega)$ (but are not necessary). This is obvious from (3.11) in the proof of the announced version of the two-energies principle.

THEOREM 3.3. *(Variant of the two-energies principle) Let $u \in H_0^2(\Omega)$ be the solution of (3.2) for $f \in L^2(\Omega)$. Moreover, for a geometrically conforming simplicial triangulation $\mathcal{T}_h(\Omega)$ of Ω let $\underline{\underline{\mathbf{p}}} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega)$ satisfy (3.9a)–(3.9c) as well as the equilibrium condition*

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{p}}} = f \quad \text{in each } K \in \mathcal{T}_h(\Omega). \tag{3.10}$$

Then, for $v \in H_0^2(\Omega)$ it holds

$$\|D^2v - \underline{\underline{\mathbf{p}}}\|_{0,\Omega}^2 = \|D^2(v-u)\|_{0,\Omega}^2 + \|D^2u - \underline{\underline{\mathbf{p}}}\|_{0,\Omega}^2.$$

Proof. Using (3.2) and applying integration by parts, we obtain

$$\int_\Omega (D^2u - \underline{\underline{\mathbf{p}}}) : D^2(u-v)\,dx = \int_\Omega f\ (u-v)\,dx - \sum_{K\in\mathcal{T}_h(\Omega)} \int_K \underline{\underline{\mathbf{p}}} : D^2(u-v)\,dx \tag{3.11}$$

$$= \sum_{K\in\mathcal{T}_h(\Omega)} \int_K (f - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{p}}})\ (u-v)\,dx - \sum_{K\in\mathcal{T}_h(\Omega)} \int_{\partial K} \underline{\underline{\mathbf{p}}}\ \mathbf{n}_{\partial K}\cdot\boldsymbol{\nabla}(u-v)\,ds$$

$$+ \sum_{K\in\mathcal{T}_h(\Omega)} \int_{\partial K} \mathbf{n}_{\partial K}\cdot\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{p}}}\ (u-v)\,ds,$$

where $\mathbf{n}_{\partial K}$ is the outward unit normal on ∂K. The first term in the second line of (3.11) vanishes due to (3.10), whereas the boundary integrals vanish due to (3.9b),(3.9c) and $u - v = \mathbf{n}_{\partial K} \cdot \boldsymbol{\nabla}(u - v) = 0$ on $\partial K \cap \Gamma$. Hence, it follows that

$$\int_\Omega (D^2 u - \underline{\underline{\mathbf{p}}}) : D^2(u - v)\, dx = 0.$$

The assertion is again an immediate consequence of this orthogonality relation. □

4. An IPDG approximation of the biharmonic equation. We consider the interior penalty discontinuous Galerkin (IPDG) approximation of the biharmonic problem (3.2) with $f \in L^2(\Omega)$ on a geometrically conforming, locally quasi-uniform simplicial triangulation $\mathcal{T}_h(\Omega)$ of the computational domain. It involves element-wise polynomial approximations of u. For $k \geq 2$ we introduce the IPDG space

$$V_h := \{v_h \in L^2(\Omega) \mid v_h|_K \in P_k(K),\ K \in \mathcal{T}_h(\Omega)\} \tag{4.1}$$

as well as the space of element-wise polynomial moment tensors

$$\underline{\underline{\mathbf{M}}}_h := \{\underline{\underline{\mathbf{q}}}_h \in L^2(\Omega)^{2\times 2} \mid \underline{\underline{\mathbf{q}}}_h|_K \in P_k(K)^{2\times 2},\ K \in \mathcal{T}_h(\Omega)\}. \tag{4.2}$$

We define a bilinear form $a_h^{IP}(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ for the variational IPDG approximation

$$\begin{aligned} a_h^{IP}(u_h, v_h) := & \sum_{K\in\mathcal{T}_h(\Omega)} \int_K D^2 u_h : D^2 v_h\, dx \qquad (4.3)\\ + & \sum_{E\in\mathcal{E}_h(\bar\Omega)} \int_E \Big(\mathbf{n}_E \cdot \{\boldsymbol{\nabla}\cdot D^2 u_h\}_E\ [v_h]_E + [u_h]_E\ \mathbf{n}_E \cdot \{\boldsymbol{\nabla}\cdot D^2 v_h\}_E\Big)\, ds \\ - & \sum_{E\in\mathcal{E}_h(\bar\Omega)} \int_E \Big([\boldsymbol{\nabla} u_h]_E \cdot \{D^2 v_h\}_E\ \mathbf{n}_E + [\boldsymbol{\nabla} v_h]_E \cdot \{D^2 u_h\}_E\ \mathbf{n}_E\Big)\, ds \\ + & \sum_{E\in\mathcal{E}_h(\bar\Omega)} \int_E \frac{\alpha_1}{h_E}\ \mathbf{n}_E \cdot [\boldsymbol{\nabla} u_h]_E\ \mathbf{n}_E \cdot [\boldsymbol{\nabla} v_h]_E\, ds + \sum_{E\in\mathcal{E}_h(\bar\Omega)} \int_E \frac{\alpha_2}{h_E^3}\ [u_h]_E\ [v_h]_E\, ds, \end{aligned}$$

where $\alpha_i > 0$, $i = 1, 2$, are suitable penalty parameters. The IPDG approximation of (3.2) reads: *Find $u_h \in V_h$ such that*

$$a_h^{IP}(u_h, v_h) = (f, v_h)_{0,\Omega}, \quad v_h \in V_h. \tag{4.4}$$

REMARK 4.1. *The Hellan–Herrmann–Johnson based symmetric IPDG approximation* (4.4) *is the counterpart of the Ciarlet–Raviart based symmetric IPDG approximation in* [27, 28]. *If we would choose the finite element space $\tilde V_h = V_h \cap C_0(\Omega)$, then it reduces to the symmetric C^0IPDG approximation considered in* [14, 15], *and* [26]. *In the C^0 case the last sum in* (4.3) *vanishes and is abandoned there.*

For completeness, we note that $a_h^{IP}(\cdot,\cdot)$ is not well defined for functions in $H_0^2(\Omega)$. This can be cured by means of a lifting operator

$$L : V_h + H_0^2(\Omega) \to \underline{\underline{\mathbf{M}}}_h$$

$$\int_\Omega L(v) : \underline{\underline{\mathbf{q}}}_h\, dx = \sum_{E\in\mathcal{E}_h(\bar\Omega)} \int_E \Big([v]_E\ \mathbf{n}_E \cdot \{\boldsymbol{\nabla}\cdot \underline{\underline{\mathbf{q}}}_h\}_E - [\boldsymbol{\nabla} v]_E \cdot \{\underline{\underline{\mathbf{q}}}_h\}_E \mathbf{n}_E\Big)\, ds. \tag{4.5}$$

The lifting operator L is stable in the sense that it satisfies (cf. [27])

$$\|L(v)\|_{0,\Omega}^2 \lesssim \sum_{E\in\mathcal{E}_h(\bar{\Omega})} \Big(h_E^{-1}\ \|n_E\cdot[\boldsymbol{\nabla} v]_E\|_{0,E}^2 + h_E^{-3}\ \|[v]_E\|_{0,E}^2\Big), \quad v\in V_h + H_0^2(\Omega).$$

Now we define $\tilde{a}_h^{IP} : (V_h + H_0^2(\Omega)) \times (V_h + H_0^2(\Omega)) \to \mathbb{R}$ as follows:

$$\begin{aligned}\tilde{a}_h^{IP}(u,v) := &\sum_{K\in\mathcal{T}_h(\Omega)} \int_K \Big(D^2u : D^2v + (L(u) : D^2v + D^2u : L(v))\Big)\,dx \\ &+ \sum_{E\in\mathcal{E}_h(\bar{\Omega})} \int_E \frac{\alpha_1}{h_E}\ \mathbf{n}_E\cdot[\boldsymbol{\nabla} u]_E\ \mathbf{n}_E\cdot[\boldsymbol{\nabla} v]_E\,ds + \sum_{E\in\mathcal{E}_h(\bar{\Omega})} \int_E \frac{\alpha_2}{h_E^3}\ [u]_E\ [v]_E\,ds.\end{aligned} \tag{4.6}$$

It is easy to verify that $\tilde{a}_h^{IP}(u_h,v_h) = a_h^{IP}(u_h,v_h)$ holds for $u_h, v_h \in V_h$.
We introduce the mesh-dependent IPDG norm on $V_h + H_0^2(\Omega)$

$$\begin{aligned}\|v\|_{2,h,\Omega}^2 := &\sum_{K\in\mathcal{T}_h(\Omega)} \|D^2v\|_{0,K}^2 \\ &+ \sum_{E\in\mathcal{E}_h(\bar{\Omega})} \frac{\alpha_1}{h_E}\ \|\mathbf{n}_E\cdot[\boldsymbol{\nabla} v]_E\|_{0,E}^2 + \sum_{E\in\mathcal{E}_h(\bar{\Omega})} \frac{\alpha_2}{h_E^3}\ \|[v]_E\|_{0,E}^2.\end{aligned} \tag{4.7}$$

It is not difficult to show that for sufficiently large penalty parameters α_i, $i=1,2$, i.e., $\alpha_1 = O((k+1)^2), \alpha_2 = O((k+1)^6)$, there exists a positive constant γ such that

$$\tilde{a}_h^{IP}(v,v) \geq \gamma\ \|v\|_{2,h,\Omega}^2, \quad v\in V_h + H_0^2(\Omega). \tag{4.8}$$

On the other hand, there exists a constant $\Gamma > 1$ such that for any $\alpha_i > 0, 1 \leq i \leq 2$,

$$\tilde{a}_h^{IP}(v,w) \leq \Gamma\ \|v\|_{2,h,\Omega}\|w\|_{2,h,\Omega}, \quad v,w\in V_h + H_0^2(\Omega). \tag{4.9}$$

In particular, it follows from (4.8) and (4.9) that the IPDG approximation (4.4) admits a unique solution $u_h \in V_h$ for sufficiently large penalty parameters.

A mixed formulation in the spirit of [6] was given in [27] for the Ciarlet–Raviart method. We provide now two mixed Hellan–Herrmann–Johnson type formulations of (4.4) by specifying appropriate numerical flux functions on the edges $E \in \mathcal{E}_h(\bar{\Omega})$

$$\underline{\widehat{\mathbf{u}}}^{(1)} := \begin{cases}\{\boldsymbol{\nabla} u_h\}_E\,, & E\in\mathcal{E}_h(\Omega)\\ 0, & E\in\mathcal{E}_h(\Gamma)\end{cases}, \tag{4.10a}$$

$$\hat{u}^{(2)} := \begin{cases}\{u_h\}_E\,, & E\in\mathcal{E}_h(\Omega)\\ 0, & E\in\mathcal{E}_h(\Gamma)\end{cases}, \tag{4.10b}$$

$$\underline{\underline{\widehat{\mathbf{p}}}} := \{D^2u_h\}_E - \frac{\alpha_1}{h_E}\ \mathbf{n}_E[\boldsymbol{\nabla} u_h]_E^T, \tag{4.10c}$$

$$\underline{\widehat{\boldsymbol{\psi}}} := \{\nabla\cdot D^2u_h\} + \frac{\alpha_2}{h_E^3}\ [u_h]_E\ \mathbf{n}_E. \tag{4.10d}$$

We keep the notion *numerical fluxes* from [6] although not all the variables in (4.10) are fluxes in the strict sense.

The mixed method with the two-field-formulation reads as follows: Find $(u_h, \underline{\underline{\mathbf{p}}}_h) \in V_h \times \underline{\underline{\mathbf{M}}}_h$ and numerical fluxes such that (4.10a)–(4.10d) holds and simultaneously for all $(v, \underline{\underline{\mathbf{q}}}) \in V_h \times \underline{\underline{\mathbf{M}}}_h$ and $K \in \mathcal{T}_h(\Omega)$

$$\int_K \underline{\underline{\mathbf{p}}}_h : \underline{\underline{\mathbf{q}}}\, dx - \int_K u_h\ \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{q}}}\, dx \tag{4.11a}$$

$$- \int_{\partial K} \widehat{\underline{\mathbf{u}}}^{(1)} \cdot \underline{\underline{\mathbf{q}}}\ \mathbf{n}_{\partial K}\, ds + \int_{\partial K} \hat{u}^{(2)} \mathbf{n}_E \cdot \boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{q}}}\, ds = 0,$$

$$\int_K \underline{\underline{\mathbf{p}}}_h : D^2 v\, dx - \int_{\partial K} \widehat{\underline{\underline{\mathbf{p}}}}\, \mathbf{n}_{\partial K} \cdot \boldsymbol{\nabla} v\, ds \tag{4.11b}$$

$$+ \int_{\partial K} \mathbf{n}_{\partial K} \cdot \widehat{\underline{\boldsymbol{\psi}}}\, v\, ds = \int_K f\ v\, dx.$$

All the equations are coupled since they contain equations on elements as well as on edges.

Often another implementation is considered as more convenient. – First the solution u_h of the primal method is determined by solving linear equations with a positive definite matrix. The numerical fluxes are determined immediately by their definition (4.10). The moment tensor $\underline{\underline{\mathbf{p}}}_h$ can be evaluated by solving the small linear system (4.11a) for each $K \in \mathcal{T}_h$.

Lemma 4.2. *Let the numerical flux functions* $\widehat{\underline{\mathbf{u}}}^{(1)}$, $\hat{u}^{(2)}$, $\widehat{\underline{\underline{\mathbf{p}}}}$ *and* $\widehat{\underline{\boldsymbol{\psi}}}$, *be given by* (4.10) *and suppose that the penalty parameters* α_i, $1 \le i \le 2$, *are sufficiently large.*

(i) *If* $u_h \in V_h$ *is the unique solution of* (4.4), *then there exists* $\underline{\underline{\mathbf{p}}}_h \in \underline{\underline{\mathbf{M}}}_h$ *such that the pair* $(u_h, \underline{\underline{\mathbf{p}}}_h)$ *satisfies* (4.11).

(ii) *If* $(u_h, \underline{\underline{\mathbf{p}}}_h) \in V_h \times \underline{\underline{\mathbf{M}}}_h$ *is a solution of* (4.11), *then* u_h *is the solution of the IPDG approximation* (4.4).

Proof. Let $u_h \in V_h$ be the unique solution of (4.4). The associated numerical fluxes are known from (4.10). We define $\underline{\underline{\mathbf{p}}}_h \in \underline{\underline{\mathbf{M}}}_h$ by means of (4.11a). Next, let $K \in \mathcal{T}_h(\Omega)$ and $v \in V_h$. We apply (4.11a) with $\underline{\underline{\mathbf{q}}}(x) = D^2 v(x)$, $x \in K$, and insert the expressions (4.10a), (4.10b) for the numerical fluxes to obtain

$$\int_K \underline{\underline{\mathbf{p}}}_h : D^2 v\, dx = \int_K u_h\ \nabla \cdot \boldsymbol{\nabla} \cdot D^2 v\, dx \tag{4.12}$$

$$+ \int_{\partial K} \{\boldsymbol{\nabla} u_h\}_{\partial K} \cdot D^2 v\ \mathbf{n}_{\partial K}\, ds - \int_{\partial K} \{u_h\}_{\partial K}\ \mathbf{n}_{\partial K} \cdot \boldsymbol{\nabla} \cdot D^2 v\, ds.$$

Using Green's formula

$$\int_K u_h\ \nabla \cdot \boldsymbol{\nabla} \cdot D_h^2 v\, dx = \int_K D_h^2 u_h : D_h^2 v\, dx \tag{4.13}$$

$$- \int_{\partial K} \boldsymbol{\nabla} u_h \cdot D^2 v\ \mathbf{n}_{\partial K}\, ds + \int_{\partial K} u_h \mathbf{n}_{\partial K} \cdot \boldsymbol{\nabla} \cdot D^2 v\, ds,$$

for eliminating the first integral on the right-hand side of (4.12) we get

$$\int_K \underline{\underline{\mathbf{p}}}_h : D^2 v \, dx \tag{4.14}$$
$$= \int_K D_h^2 u_h : D_h^2 v \, dx - \int_{\partial K} \boldsymbol{\nabla} u_h \cdot D^2 v \ \mathbf{n}_{\partial K} \, ds + \int_{\partial K} u_h \mathbf{n}_{\partial K} \cdot \boldsymbol{\nabla} \cdot D^2 v \, ds$$
$$+ \int_{\partial K} \{\boldsymbol{\nabla} u_h\}_{\partial K} \cdot D^2 v \ \mathbf{n}_{\partial K} \, ds - \int_{\partial K} \{u_h\}_{\partial K} \ \mathbf{n}_{\partial K} \cdot \boldsymbol{\nabla} \cdot D^2 v \, ds.$$

We recall that $\{w\}_{\partial K} - w|_{\partial K} = \frac{1}{2}[w]_{\partial K}$. Summation over all triangles yields

$$\sum_{K \in \mathcal{T}_h(\Omega)} \int_K \underline{\underline{\mathbf{p}}}_h : D^2 v \, dx = \sum_{K \in \mathcal{T}_h(\Omega)} \int_K D_h^2 u_h : D_h^2 v \, dx \tag{4.15}$$
$$+ \sum_{E \in \mathcal{E}_h(\Omega)} \int_E [\boldsymbol{\nabla} u_h]_E \cdot \{D^2 v\}_E \mathbf{n}_E \, ds - \sum_{E \in \mathcal{E}_h(\Omega)} \int_E [u_h]_E \ \mathbf{n}_E \cdot \{\boldsymbol{\nabla} \cdot D^2 v\}_E \, ds.$$

Next, we use the variational equality (4.4) to eliminate the first integral on the right.hand side of (4.15),

$$\sum_{K \in \mathcal{T}_h(\Omega)} \int_K \underline{\underline{\mathbf{p}}}_h : D^2 v \, dx \tag{4.16}$$
$$= \sum_{E \in \mathcal{E}_h(\bar{\Omega})} \int_E \Big(\{\mathbf{n}_E \cdot \boldsymbol{\nabla} \cdot D^2 u_h\}_E \ [v]_E + [u_h]_E \ \mathbf{n}_E \cdot \{\boldsymbol{\nabla} \cdot D^2 v\}_E \Big) \, ds$$
$$- \sum_{E \in \mathcal{E}_h(\bar{\Omega})} \int_E \Big([\boldsymbol{\nabla} u_h]_E \cdot \{D^2 v \ \mathbf{n}_E\}_E + \{D^2 u_h \ \mathbf{n}_E\}_E \cdot [\boldsymbol{\nabla} v]_E \Big) \, ds$$
$$- \sum_{E \in \mathcal{E}_h(\bar{\Omega})} \int_E \frac{\alpha_1}{h_E} \ [\mathbf{n}_E \cdot \boldsymbol{\nabla} u_h]_E \ [\mathbf{n}_E \cdot \boldsymbol{\nabla} v]_E \, ds - \sum_{E \in \mathcal{E}_h(\bar{\Omega})} \int_E \frac{\alpha_2}{h_E^3} \ [u_h]_E \ [v]_E \, ds$$
$$+ \int_\Omega f v \, dx$$
$$+ \sum_{E \in \mathcal{E}_h(\Omega)} \int_E [\boldsymbol{\nabla} u_h]_E \cdot \{D^2 v\}_E \mathbf{n}_E \, ds - \sum_{E \in \mathcal{E}_h(\Omega)} \int_E [u_h]_E \ \mathbf{n}_E \cdot \boldsymbol{\nabla} \cdot \{D^2 v\}_E \, ds.$$

Note that four integrals in (4.16) cancel. Observing (4.10c),(4.10d) we obtain (4.11b). Conversely, if $(u_h, \underline{\underline{\mathbf{p}}}_h) \in V_h \times \underline{\underline{\mathbf{M}}}_h$ solves (4.11a), (4.11b), we choose $\underline{\underline{q}} := D^2 v$ in (4.11a). Applying Green's formula (4.13) again, we can eliminate $\underline{\underline{\mathbf{p}}}_h$ from the system. It follows that u_h is a solution of the primal problem (4.4) which proves (ii). ◻

Instead of the two-field formulation (4.11) we consider next a three-field formulation by introducing the finite element space

$$\underline{\mathbf{W}}_h := \{\underline{\boldsymbol{\phi}}_h \in L^2(\Omega)^2 \mid \underline{\boldsymbol{\phi}}_h|_K \in P_{k-1}(K)^2, K \in \mathcal{T}_h(\Omega)\}. \tag{4.17}$$

The three-field formulation reads as follows: Find $(u_h, \underline{\underline{\mathbf{p}}}_h, \underline{\boldsymbol{\psi}}_h) \in V_h \times \underline{\underline{\mathbf{M}}}_h \times \underline{\mathbf{W}}_h$ together with the numerical flux functions $\widehat{\underline{\mathbf{u}}}^{(1)}$, $\hat{u}^{(2)}$, $\widehat{\underline{\underline{\mathbf{p}}}}$ and $\widehat{\underline{\boldsymbol{\psi}}}$ in (4.10) such that for all $(v, \underline{\underline{\mathbf{q}}}, \underline{\boldsymbol{\phi}}) \in V_h \times \underline{\underline{\mathbf{M}}}_h \times \underline{\mathbf{W}}_h$ and all $K \in \mathcal{T}_h(\Omega)$ it holds

$$\int_K \underline{\underline{\mathbf{p}}}_h : \underline{\underline{\mathbf{q}}}\, dx - \int_K u_h\ \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{q}}}\, dx \tag{4.18a}$$

$$- \int_{\partial K} \widehat{\underline{\mathbf{u}}}^{(1)} \cdot \underline{\underline{\mathbf{q}}}\, \mathbf{n}_{\partial K}\, ds + \int_{\partial K} \hat{u}^{(2)} \mathbf{n}_{\partial K} \cdot \boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{q}}}\, ds = 0,$$

$$\int_K \underline{\underline{\mathbf{p}}}_h : \boldsymbol{\nabla}\underline{\boldsymbol{\phi}}\, dx - \int_{\partial K} \widehat{\underline{\underline{\mathbf{p}}}}\, \mathbf{n}_{\partial K} \cdot \underline{\boldsymbol{\phi}}\, ds = - \int_K \underline{\boldsymbol{\psi}}_h \cdot \underline{\boldsymbol{\phi}}\, dx, \tag{4.18b}$$

$$\int_K \underline{\boldsymbol{\psi}}_h \cdot \boldsymbol{\nabla} v\, dx - \int_{\partial K} \mathbf{n}_{\partial K} \cdot \widehat{\underline{\boldsymbol{\psi}}}\, v\, ds = - \int_K f v\, dx. \tag{4.18c}$$

LEMMA 4.3. *Under the assumptions of Lemma 4.2 it holds:*

(i) *If $u_h \in V_h$ is the unique solution of* (4.4)*, then there exists a unique pair $(\underline{\underline{\mathbf{p}}}_h, \underline{\boldsymbol{\psi}}_h) \in \underline{\underline{\mathbf{M}}}_h \times \underline{\mathbf{W}}_h$ such that the triple $(u_h, \underline{\underline{\mathbf{p}}}_h, \underline{\boldsymbol{\psi}}_h)$ satisfies* (4.18)*.*

(ii) *If $(u_h, \underline{\underline{\mathbf{p}}}_h, \underline{\boldsymbol{\psi}}_h) \in V_h \times \underline{\underline{\mathbf{M}}}_h \times \underline{\mathbf{W}}_h$ is a solution of* (4.18)*, then the pair $(u_h, \underline{\underline{\mathbf{p}}}_h)$ solves* (4.11)*, and u_h is the solution of the IPDG approximation* (4.4)*.*

Proof. If $u_h \in V_h$ is the unique solution of (4.4), we already know from Lemma 4.2(i) that there exists $\underline{\underline{\mathbf{p}}}_h \in \underline{\underline{\mathbf{M}}}_h$ such that (4.11a) and (4.11b) are satisfied. Next, we define $\underline{\boldsymbol{\psi}}_h \in \underline{\mathbf{W}}_h$ by means of (4.18b). Choosing $\underline{\boldsymbol{\phi}} = \boldsymbol{\nabla} v$ we may replace the first two terms in (4.11b) by $\sum_K \int_K \underline{\boldsymbol{\psi}}_h \cdot \boldsymbol{\nabla} v\, dx$. It follows that (4.18c) holds true which proves (i).

Conversely, if $(u_h, \underline{\underline{\mathbf{p}}}_h, \underline{\boldsymbol{\psi}}_h) \in V_h \times \underline{\underline{\mathbf{M}}}_h \times \underline{\mathbf{W}}_h$ is a solution of (4.18a)–(4.18c), obviously (4.11a) and (4.18a) coincide. Next, we set $\underline{\boldsymbol{\phi}} = \boldsymbol{\nabla} v$ in (4.18b) and evaluate the term in the second line via (4.18c),

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h(\Omega)} \int_K \underline{\underline{\mathbf{p}}}_h : D^2 v\, dx - \sum_{K\in\mathcal{T}_h(\Omega)} \int_{\partial K} \widehat{\underline{\underline{\mathbf{p}}}}\, \mathbf{n}_{\partial K} \cdot \boldsymbol{\nabla} v\, ds \\
&= - \sum_{K\in\mathcal{T}_h(\Omega)} \int_K \underline{\boldsymbol{\psi}}_h \cdot \boldsymbol{\nabla} v\, dx, \\
&= - \sum_{K\in\mathcal{T}_h(\Omega)} \int_{\partial K} \mathbf{n}_{\partial K} \cdot \widehat{\underline{\boldsymbol{\psi}}}\, v\, ds + \sum_{K\in\mathcal{T}_h(\Omega)} \int_K f v\, dx.
\end{aligned}$$

Hence, we obtain (4.11b). Now Lemma 4.2, part (ii) shows that u_h solves (4.4) which proves (ii). □

5. An a posteriori error estimator for the IPDG approximation of the biharmonic equation. The construction of an equilibrated moment tensor in the finite element framework will be affected by data oscillation, and the case $k = 2$ requires special care. This will be clear from Remark 6.5 below. Specifically, set

$$\underline{\underline{\mathbf{M}}}_h^{eq} := \{\underline{\underline{\mathbf{q}}}_h \in L^2(\Omega)^{2\times 2} \mid \underline{\underline{\mathbf{q}}}_h|_K \in P_\ell(K)^{2\times 2},\ K \in \mathcal{T}_h(\Omega)\}, \tag{5.1}$$

$$\text{where}\quad \ell := \begin{cases} k & \text{if } k \geq 3, \\ 3 & \text{if } k = 2. \end{cases}$$

Given $K \in \mathcal{T}_h(\Omega)$, let f_K be the L^2-projection of f onto $P_{\ell-2}(K)$, and let $f_h \in L^2(\Omega)$ be given by $f_h|_K = f_K, K \in \mathcal{T}_h(\Omega)$. A moment tensor $\underline{\underline{\mathbf{p}}}_h^{eq} \in \underline{\underline{\mathbf{M}}}_h^{eq}$ is called equilibrated in this framework, if if satisfies (3.9b),(3.9c) which implies $\underline{\underline{\mathbf{p}}}_h^{eq} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega)$, and also the equilibrium equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{p}}}_h^{eq} = f_h \quad \text{in each } K \in \mathcal{T}_h(\Omega). \tag{5.2}$$

The two-energies principle (Theorem 3.3) can be applied to the IPDG approximation (4.4) involving an equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_h^{eq}$. It gives rise to an a posteriori error bound in terms of element-related terms $\eta_{K,i}^{eq}, 1 \le i \le 2$, and edge-related terms $\eta_{E,i}^{eq}, 1 \le i \le 2$, as given by

$$\eta_{K,1}^{eq} := \|D^2 u_h - \underline{\underline{\mathbf{p}}}_h^{eq}\|_{0,K}, \quad K \in \mathcal{T}_h(\Omega), \tag{5.3a}$$

$$\eta_{K,2}^{eq} := \|D^2 u_h - D^2 u_h^{conf}\|_{0,K}, \quad K \in \mathcal{T}_h(\Omega), \tag{5.3b}$$

$$\eta_{E,1}^{eq} := h_E^{-1/2} \ \|\mathbf{n}_E \cdot [\boldsymbol{\nabla} u_h]_E\|_{0,E}, \quad E \in \mathcal{E}_h(\bar{\Omega}), \tag{5.3c}$$

$$\eta_{E,2}^{eq} := h_E^{-3/2} \ \|[u_h]_E\|_{0,E}, \quad E \in \mathcal{E}_h(\bar{\Omega}), \tag{5.3d}$$

where $u_h^{conf} \in H_0^2(\Omega)$ in (5.3b) will be constructed by postprocessing from the finite element solution $u_h \in V_h$.

The following auxiliary result deals with the data oscillation due to the approximation of f by f_h. Its application is not restricted to a posteriori error estimates.

Lemma 5.1. *Let $z \in H_0^2(\Omega)$ be the weak solution of the biharmonic problem*

$$\Delta^2 z = f - f_h \quad \textit{in } \Omega, \tag{5.4a}$$

$$z = \mathbf{n}_\Gamma \cdot \boldsymbol{\nabla} z = 0 \qquad \textit{on } \Gamma = \partial\Omega. \tag{5.4b}$$

If the L^2-projection of $f - f_h$ to $P_1(K)$ in each $K \in \mathcal{T}_h$ vanishes, then

$$\|D^2 z\|_{0,\Omega}^2 \le C \sum_{K \in \mathcal{T}_h(\Omega)} h_K^4 \ \|f - f_h\|_{0,K}^2. \tag{5.5}$$

Proof. For $v \in H_0^2(\Omega)$ and $p_1 \in P_1(K), K \in \mathcal{T}_h(\Omega)$, we have by assumption

$$\sum_{K \in \mathcal{T}_h(\Omega)} (D^2 z, D^2 v)_{0,K} = \sum_{K \in \mathcal{T}_h(\Omega)} (f - f_h, v - p_1)_{0,K}.$$

Choosing $v = z$, it follows that

$$\sum_{K \in \mathcal{T}_h(\Omega)} \|D^2 z\|_{0,K}^2 \le \sum_{K \in \mathcal{T}_h(\Omega)} \|f - f_h\|_{0,K} \ \|z - p_1\|_{0,K}.$$

We fix $p_1 \in P_1(K)$ by the interpolation conditions $\int_K p_1 dx = \int_K z dx$ and $\int_K \boldsymbol{\nabla} p_1 dx = \int_K \boldsymbol{\nabla} z dx$. The Poincaré-Friedrichs inequalities (cf., e.g., [34])

$$\|z - p_1\|_{0,K} \le C h_K \Big(\|\boldsymbol{\nabla} z\|_{0,K} + \Big| \int_K (z - p_1) \, dx \Big| \Big),$$

$$\|\boldsymbol{\nabla}(z - p_1)\|_{0,K} \le C h_K \Big(\|D^2 z\|_{0,K} + \Big| \int_K \boldsymbol{\nabla}(z - p_1) \, dx \ \Big|_{\mathbb{R}^2} \Big).$$

yield the relation

$$\sum_{K\in\mathcal{T}_h(\Omega)} \|D^2 z\|_{0,K}^2 \le C^2 \sum_{K\in\mathcal{T}_h(\Omega)} \|D^2 z\|_{0,K} h_K^2 \|f-f_h\|_{0,K}\,.$$

By applying the Cauchy inequality to the right-hand side and dividing by the square root of the left-hand side we obtain the assertion. ∎

The data oscillation will be denoted by

$$osc_h^2(f) := \sum_{K\in\mathcal{T}_h(\Omega)} osc_K^2(f), \quad osc_K^2(f) := h_K^4\ \|f-f_h\|_{0,K}^2. \tag{5.6}$$

The error bound in the following theorem refers to the norm (4.7).

THEOREM 5.2. *Let $u\in H_0^2(\Omega)$ be the solution of the biharmonic problem* (3.1a), (3.1b)*, let $u_h\in V_h$ be the unique solution of the IPDG approximation* (4.4)*, and let $\underline{\underline{\mathbf{p}}}_h^{eq}\in \underline{\underline{\mathbf{M}}}_h^{eq}\cap \underline{\underline{\mathbf{H}}}(\mathrm{div}^2,\Omega)$ be an equilibrated moment tensor. Moreover, let $u_h^{conf}\in H_0^2(\Omega)$, let $\eta_{K,i}^{eq},\eta_{E,i}^{eq}, 1\le i\le 2$, be given by* (5.3a)–(5.3d)*, and let $osc_h(f)$ be the data oscillation* (5.6)*. We set*

$$\begin{aligned}\eta_h^{eq} := &\Big(\sum_{K\in\mathcal{T}_h(\Omega)} (\eta_{K,1}^{eq})^2\Big)^{1/2} + 2\Big(\sum_{K\in\mathcal{T}_h(\Omega)} (\eta_{K,2}^{eq})^2\Big)^{1/2} \\ &+ \Big(\sum_{E\in\mathcal{E}_h(\bar\Omega)} (\alpha_1(\eta_{E,1}^{eq})^2+\alpha_2(\eta_{E,2}^{eq})^2)\Big)^{1/2}.\end{aligned} \tag{5.7}$$

Then there exists a constant $C_1>0$, which only depends on the local geometry of the triangulation, such that it holds

$$\|u-u_h\|_{2,h,\Omega} \le \eta_h^{eq} + C_1\ osc_h(f). \tag{5.8}$$

Proof. Let $\bar u\in H_0^2(\Omega)$ be the weak solution of the biharmonic problem

$$\begin{aligned}\Delta^2 \bar u &= f_h \quad \text{in } \Omega,\\ \bar u = \mathbf{n}_\Gamma\cdot\boldsymbol{\nabla}\bar u &= 0 \quad \text{on } \Gamma=\partial\Omega.\end{aligned}$$

By recalling (4.7) and applying the triangle inequality twice we obtain

$$\begin{aligned}&\|u-u_h\|_{2,h,\Omega} \\ &\le \Big(\sum_{K\in\mathcal{T}_h(\Omega)} \|D^2u-D^2u_h\|_{0,K}^2\Big)^{1/2} + \Big(\sum_{E\in\mathcal{E}_h(\bar\Omega)} (\alpha_1(\eta_{E,1}^{eq})^2+\alpha_2(\eta_{E,2}^{eq})^2)\Big)^{1/2} \\ &\le \Big(\sum_{K\in\mathcal{T}_h(\Omega)} \|D^2u-D^2\bar u\|_{0,K}^2\Big)^{1/2} + \Big(\sum_{K\in\mathcal{T}_h(\Omega)} \|D^2\bar u-D^2u_h^{conf}\|_{0,K}^2\Big)^{1/2} \\ &+ \Big(\sum_{K\in\mathcal{T}_h(\Omega)} \|D^2u_h^{conf}-D^2u_h)\|_{0,K}^2\Big)^{1/2} + \Big(\sum_{E\in\mathcal{E}_h(\bar\Omega)} (\alpha_1(\eta_{E,1}^{eq})^2+\alpha_2(\eta_{E,2}^{eq})^2)\Big)^{1/2}.\end{aligned} \tag{5.9}$$

Since $z:=u-\bar u$ solves (5.4a),(5.4b), the first term in the third line of (5.9) can be estimated from above by Lemma 5.1 and thus gives rise to the data oscillation term in

(5.8). The two-energies principle (Theorem 3.3) with $u=\bar u$, $v=u_h^{conf}$ and $\underline{\underline{\mathbf{p}}}=\underline{\underline{\mathbf{p}}}_h^{eq}$ yields

$$\|D^2(\bar u-u_h^{conf})\|_{0,\Omega}\le\Big(\sum_{K\in\mathcal{T}_h(\Omega)}\|D^2u_h^{conf}-\underline{\underline{\mathbf{p}}}_h^{eq}\|_{0,K}^2\Big)^{1/2}\le \tag{5.10}$$
$$\Big(\sum_{K\in\mathcal{T}_h(\Omega)}\Big(\|D^2u_h-D^2u_h^{conf}\|_{0,K}^2\Big)^{1/2}+\Big(\sum_{K\in\mathcal{T}_h(\Omega)}\|\underline{\underline{\mathbf{p}}}_h^{eq}-D^2u_h\|_{0,K}^2\Big)^{1/2}.$$

Using these estimates in (5.9) allows to conclude. □

Remark 5.3. *We note that the constant C_1 in front of the data oscillation term $osc_h(f)$ is the only generic constant occurring in the reliability estimate* (5.8).

In practice, a modified equilibrated error estimator avoids the computationally expensive evaluation of u_h^{conf} and attracts attention, although the reliability estimate (5.11) below contains another generic constant.

Corollary 5.4. *Assume that the assumptions of* Theorem 5.2 *are satisfied. Specifically, let V_h^{conf} be the generalized version of the classical Hsieh–Clough–Tocher C^1 conforming finite element space as constructed in* [20]*, and let $u_h^{conf}=E_h(u_h)$ be the extension of u_h to V_h^{conf} as defined in* [28]*. Then there exists a constant $C_2>0$, depending only on the local geometry of the triangulation and on the penalty parameters $\alpha_i, 1\le i\le 2$, such that it holds*

$$\|u-u_h\|_{2,h,\Omega}\ \le \tag{5.11}$$
$$\Big(\sum_{K\in\mathcal{T}_h(\Omega)}(\eta_{K,1}^{eq})^2\Big)^{1/2}+C_2\Big(\sum_{E\in\mathcal{E}_h(\bar\Omega)}((\eta_{E,1}^{eq})^2+(\eta_{E,2}^{eq})^2)\Big)^{1/2}+C\ osc_h(f).$$

Proof. In [28] it has been shown that

$$\sum_{K\in\mathcal{T}_h(\Omega)}(\eta_{K,2}^{eq})^2\lesssim\sum_{E\in\mathcal{E}_h(\bar\Omega)}((\eta_{E,1}^{eq})^2+(\eta_{E,2}^{eq})^2). \tag{5.12}$$

Using (5.12) in (5.8) yields (5.11). □

6. Construction of an equilibrated moment tensor. We construct an equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_h^{eq}\in\underline{\underline{\mathbf{M}}}_h^{eq}\cap\underline{\underline{\mathbf{H}}}(\mathrm{div}^2,\Omega)$ which allows to apply the two-energies principle and Theorem 5.2. The construction will be done by an interpolation on each element. Thus it is a local procedure. In particular, denoting by $\mathbf{BDM}_m(K), m\in\mathbb{N}$, the Brezzi-Douglas-Marini element of polynomial degree m (cf., e.g., [16]), we first construct an auxiliary vector field $\underline{\boldsymbol{\psi}}_h^{eq}\in\mathbf{H}(\mathrm{div},\Omega), \underline{\boldsymbol{\psi}}_h^{eq}|_K\in\mathbf{BDM}_{\ell-1}(K), K\in\mathcal{T}_h(\Omega)$, satisfying

$$\boldsymbol{\nabla}\cdot\underline{\boldsymbol{\psi}}_h^{eq}=f_h\quad\text{in } L^2(\Omega), \tag{6.1}$$

and then an equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_h^{eq}\in\underline{\underline{\mathbf{M}}}_h^{eq}\cap\underline{\underline{\mathbf{H}}}(\mathrm{div}^2,\Omega)$ satisfying

$$\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{p}}}_h^{eq}=\underline{\boldsymbol{\psi}}_h^{eq}\quad\text{in } L^2(\Omega)^2. \tag{6.2}$$

For the construction of the auxiliary vector field we recall the following result:

LEMMA 6.1. *Let $m \geq 1$. Any vector field $\underline{\phi} \in P_m(K)$ is uniquely defined by the following degrees of freedom*

$$\int_E \mathbf{n}_E \cdot \underline{\phi}\, q\, ds, \quad q \in P_m(E),\ E \in \mathcal{E}_h(\partial K), \tag{6.3a}$$

$$\int_K \underline{\phi} \cdot \boldsymbol{\nabla} q\, dx, \qquad q \in P_{m-1}(K), \tag{6.3b}$$

$$\int_K \underline{\phi} \cdot \mathbf{curl}(b_K q)\, dx, \quad q \in P_{m-2}(K). \tag{6.3c}$$

where b_K in (6.3c) *is the element bubble function on K given by $b_K = \prod_{i=1}^{3} \lambda_i^K$ and $\lambda_i^K, 1 \leq i \leq 3$, are the barycentric coordinates of K. Moreover, there exists a positive constant $C_1(m)$ depending only on the polynomial degree m and the local geometry of the triangulation $\mathcal{T}_h(\Omega)$ such that*

$$\int_K |\underline{\phi}|^2\, dx \leq C_1(k) \Big(\sum_{E \in \mathcal{E}_h(\partial K)} h_E \int_E |\mathbf{n}_E \cdot \underline{\phi}|^2\, ds + h_K^2 \int_K |\boldsymbol{\nabla} \cdot \underline{\phi}|^2\, dx \tag{6.4}$$
$$+\ h_K^2 \max \Big\{ \int_K |\underline{\phi} \cdot \mathbf{curl}(b_K q)|^2\, dx;\ q \in P_{m-2}(K),\ \max_{x \in K} |q(x)| \leq 1 \Big\} \Big).$$

Proof. For the uniqueness result we refer to (3.41) in [16, p. 125] since $\mathbf{BDM}_m(K) = P_m(K)$. The estimate (6.4) can be derived by standard scaling arguments (cf. Lemma 3.1 and Remark 3.3 in [9]). ∎

The auxiliary vector field $\underline{\psi}_h^{eq}$ is constructed in each element $K \in \mathcal{T}_h$ such that $\underline{\psi}_h^{eq}|_K \in \mathbf{BDM}_{\ell-1}(K)$ satisfies the interpolation conditions

$$\int_E \mathbf{n}_E \cdot \underline{\psi}_h^{eq}\, q\, ds = \int_E \mathbf{n}_E \cdot \widehat{\underline{\psi}}\, q\, ds, \quad q \in P_{\ell-1}(E),\ E \in \mathcal{E}_h(\partial K), \tag{6.5a}$$

$$\int_K \underline{\psi}_h^{eq} \cdot \boldsymbol{\nabla} q\, dx = \int_K \underline{\psi}_h \cdot \boldsymbol{\nabla} q\, dx, \qquad q \in P_{\ell-2}(K), \tag{6.5b}$$

$$\int_K \underline{\psi}_h^{eq} \cdot \mathbf{curl}(b_K q)\, dx = \int_K \boldsymbol{\nabla} \cdot D^2 u_h \cdot \mathbf{curl}(b_K q)\, dx, \quad q \in P_{\ell-3}(K). \tag{6.5c}$$

LEMMA 6.2. *The vector field $\underline{\psi}_h^{eq}$ that is defined by* (6.5) *is contained in $\mathbf{H}(\mathrm{div}, \Omega)$ and satisfies* (6.1).

Proof. The solvability of (6.5a)–(6.5c) is guaranteed by Lemma 6.1 with $m = \ell - 1$. The continuity of the normal components follows from (6.5a) on adjacent triangles and yields $\underline{\psi}_h^{eq} \in \mathbf{H}(\mathrm{div}, \Omega)$.

Let $K \in \mathcal{T}_h(\Omega)$. Given a polynomial $q \in P_{\ell-2} \subset P_k$, we can use (4.18c) with $v|_K = q$ and $v_h|_{K'} = 0,\ K \neq K' \in \mathcal{T}_h(\Omega)$. Moreover we make use of Green's formula, as well as of (6.5a) and (6.5b) to obtain

$$\begin{aligned}\int_K \boldsymbol{\nabla}\cdot \underline{\boldsymbol{\psi}}_h^{eq} q\,dx &= -\int_K \underline{\boldsymbol{\psi}}_h^{eq}\cdot \boldsymbol{\nabla} q\,dx + \int_{\partial K} \mathbf{n}_{\partial K}\cdot \underline{\boldsymbol{\psi}}_h^{eq} q\,ds\\ &= -\int_K \underline{\boldsymbol{\psi}}_h \cdot \boldsymbol{\nabla} q\,dx + \int_{\partial K} \mathbf{n}_{\partial K}\cdot \widehat{\underline{\boldsymbol{\psi}}}\, q\,ds\\ &= \int_K f q\,dx = \int_K f_h q\,dx.\end{aligned}$$

Since both $\boldsymbol{\nabla}\cdot \underline{\boldsymbol{\psi}}_h^{eq}$ and f_h live in $P_{\ell-2}(K)$, (6.1) follows from the preceding equation. Now, the assertion follows from $\underline{\boldsymbol{\psi}}_h^{eq} \in \mathbf{H}(\mathrm{div},\Omega)$. ∎

The construction (6.5) by local interpolation and Lemma 6.2 take into account that there is a compatibility condition due to Gauss' theorem. The divergence of $\underline{\boldsymbol{\psi}}_h^{eq}$ in K cannot be fixed independently of the normal components of $\underline{\boldsymbol{\psi}}_h^{eq}$ on ∂K, but the latter are required in order to achieve the continuity of the normal components and $\underline{\boldsymbol{\psi}}_h^{eq} \in \mathbf{H}(\mathrm{div},\Omega)$.
The compatibility conditions are satisfied here due to the finite element equation (4.18c) for the discontinuous Galerkin (IPDG) method. They enable us to proceed on elements like e.g., in [9, 18, 22], and we need not operate on patches like in the applications of the two-energies principle and H^1-conforming elements as, e.g., in [10, 12] or [8, Section III.9].

For the construction of the equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_h^{eq}$ we begin with the specification of the degrees of freedom for tensors $\underline{\underline{\mathbf{p}}} \in P_\ell(K)^{2\times 2}$.

Lemma 6.3. *We have dim* $P_\ell(K)^{2\times 2} = 2(\ell+1)(\ell+2)$. *Any* $\underline{\underline{\mathbf{p}}} \in P_\ell(K)^{2\times 2}$, $\underline{\underline{\mathbf{p}}} = (p_{ij})_{i,j=1}^2$, *with* $\underline{\mathbf{p}}^{(i)} := (p_{i1},p_{i2})^T, 1\le i\le 2$, *is uniquely determined by the following degrees of freedom (DOF)*

$$\int_E (\underline{\underline{\mathbf{p}}}\ \mathbf{n}_E)\cdot \underline{\mathbf{q}}\,ds, \qquad \underline{\mathbf{q}} \in P_\ell(E)^2,\ E\in\mathcal{E}_h(\partial K)., \tag{6.6a}$$

$$\int_K \underline{\underline{\mathbf{p}}} : \nabla \underline{\mathbf{q}}\,dx, \qquad \underline{\mathbf{q}} \in P_{\ell-1}(K)^2\backslash P_0(K)^2, \tag{6.6b}$$

$$\int_K \underline{\mathbf{p}}^{(i)}\cdot \mathbf{curl}(b_K q)\,dx, \quad q\in P_{\ell-2}(K),\ 1\le i\le 2. \tag{6.6c}$$

The numbers of degrees of freedom (DOF) associated with (6.6a)–(6.6c) *are as follows*

$$\begin{aligned} &\mathit{DOF}\ (6.6\mathrm{a}) = 6(\ell+1),\\ &\mathit{DOF}\ (6.6\mathrm{b}) = \ell(\ell+1)-2,\\ &\mathit{DOF}\ (6.6\mathrm{c}) = \ell(\ell-1)\end{aligned}$$

and sum up to $2(\ell+1)(\ell+2)$.

Proof. The interpolation conditions for $\underline{\mathbf{p}}^{(1)}$ and $\underline{\mathbf{p}}^{(2)}$ are separated. The vector field

$\underline{\mathbf{p}}^{(i)}$ (for $1 \le i \le 2$) is determined by the degrees of freedom

$$\begin{aligned} &\int_E \underline{\mathbf{p}}^{(i)}\, \mathbf{n}_E\, q\, ds, && q \in P_\ell(E),\ E \in \mathcal{E}_h(\partial K)., \\ &\int_K \underline{\mathbf{p}}^{(i)} \cdot \nabla q\, dx, && q \in P_{\ell-1}(K)\backslash P_0(K), \\ &\int_K \underline{\mathbf{p}}^{(i)} \cdot \mathbf{curl}(b_K q)\, dx, && q \in P_{\ell-2}(K)\,. \end{aligned}$$

By applying Lemma 6.1 with $m = \ell$ we conclude that there is a unique solution. □

Lemma 6.4. *Let $\underline{\underline{\mathbf{q}}} = (\underline{\mathbf{q}}^{(1)}, \underline{\mathbf{q}}^{(2)}) \in P_\ell(K)^{2\times 2}$. Then there exists a positive constant $C_2(\ell)$ depending only on the polynomial degree ℓ and the local geometry of the triangulation $\mathcal{T}_h(\Omega)$ such that*

$$\begin{aligned} \int\limits_K |\underline{\underline{\mathbf{q}}}|^2\, dx \le C_2(k)\, \Big(&\sum_{E \in \mathcal{E}_h(\partial K)} h_E \int\limits_E (|\mathbf{n}_E \cdot \underline{\underline{\mathbf{q}}}\mathbf{n}_E|^2 + |\mathbf{t}_E \cdot \underline{\underline{\mathbf{q}}}\mathbf{n}_E|^2)\, ds \\ &+ h_K^2 \int\limits_K |\boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{q}}}|^2\, dx \\ &+\ h_K^2 \sum_{i=1}^2 \max\Big\{ \int\limits_K |\underline{\mathbf{q}}^{(i)} \cdot \mathbf{curl}(b_K q_{\ell-2})|^2\, dx;\ q_{\ell-2} \in P_{\ell-2}, \max_{x\in K} |q_{k-2}(x)| \le 1 \Big\}\Big). \end{aligned} \tag{6.7}$$

Proof. As in the proof of Lemma 6.1, the estimate (6.7) follows by standard scaling arguments. □

Now, for the construction of the equilibrated moment tensor we set $\underline{\underline{\mathbf{z}}}_h := D^2 u_h$ with

$$\underline{\mathbf{z}}_h^{(1)} := (\frac{\partial^2 u_h}{\partial x_1^2}, \frac{\partial^2 u_h}{\partial x_1 \partial x_2})^T, \quad \underline{\mathbf{z}}_h^{(2)} := (\frac{\partial^2 u_h}{\partial x_1 \partial x_2}, \frac{\partial^2 u_h}{\partial x_2^2})^T.$$

We construct $\underline{\underline{\mathbf{p}}}_h^{eq} = (p_{ij}^{h,eq})_{i,j=1}^2$, with $\underline{\mathbf{p}}_{h,eq}^{(i)} = (p_{i1}^{h,eq}, p_{i2}^{h,eq})^T, 1 \le i \le 2$, in each element K by fixing the degrees of freedom (6.6a)–(6.6c) according to

$$\int\limits_E \underline{\underline{\mathbf{p}}}_h^{eq}\, \mathbf{n}_E \cdot \underline{\mathbf{q}}\, ds = \int\limits_E \widehat{\underline{\underline{\mathbf{p}}}}\, \mathbf{n}_E \cdot \underline{\mathbf{q}}\, ds, \quad \underline{\mathbf{q}} \in P_\ell(E)^2,\ E \in \mathcal{E}_h(\partial K), \tag{6.8a}$$

$$\int\limits_K \underline{\underline{\mathbf{p}}}_h^{eq} : \boldsymbol{\nabla}\underline{\mathbf{q}}\, dx \ = -\int\limits_K \underline{\boldsymbol{\psi}}_h^{eq} \cdot \underline{\mathbf{q}}\, dx + \int\limits_{\partial K} \widehat{\underline{\underline{\mathbf{p}}}}\, \mathbf{n}_{\partial K} \cdot \underline{\mathbf{q}}\, ds, \quad \underline{\mathbf{q}} \in P_{\ell-1}(K)^2 \tag{6.8b}$$

$$\int\limits_K \underline{\mathbf{p}}_{h,eq}^{(i)} \cdot \mathbf{curl}(b_K q)\, dx \ = \int\limits_K \underline{\mathbf{z}}_h^{(i)} \cdot \mathbf{curl}(b_K q)\, dx, \quad q \in P_{\ell-2}(K),\ 1 \le i \le 2. \tag{6.8c}$$

Remark 6.5. *Obviously, the equations* (6.8b) *require the compatibility conditions*

$$-\int_K \underline{\boldsymbol{\psi}}_h^{eq} \cdot \underline{\mathbf{p}}\, dx + \int_{\partial K} \widehat{\underline{\underline{\mathbf{p}}}}\, \mathbf{n}_{\partial K} \cdot \underline{\mathbf{p}}\, ds = 0, \qquad \underline{\mathbf{p}} \in P_0(K)^2 \tag{6.9}$$

with constant polynomials $\underline{\mathbf{p}} \in P_0(K)^2$. *Indeed, we had to care for* $\ell \geq 3$ *in* (5.1) *in order to verify* (6.9) *now. From the finite element equation* (4.18b) *we conclude that*

$$-\int_K \underline{\boldsymbol{\psi}}_h \cdot \underline{\mathbf{p}}\, dx + \int_{\partial K} \underline{\underline{\widehat{\mathbf{p}}}}\, \mathbf{n}_{\partial K} \cdot \underline{\mathbf{p}}\, ds = 0, \qquad \underline{\mathbf{p}} \in P_0(K)^2.$$

Given $\underline{\mathbf{p}} = (p_1, p_2) \in P_0(K)^2$, *there exists* $q \in P_1(K)$ *with* $\underline{\mathbf{p}} = \nabla q$, *specifically* $(p_1, p_2) = \nabla(p_1 x_1 + p_2 x_2)$. *Since* $\ell \geq 3$, *we conclude from* (6.5b) *that*

$$\int_K \underline{\boldsymbol{\psi}}_h^{eq} \cdot \underline{\mathbf{p}}\, dx = \int_K \underline{\boldsymbol{\psi}}_h^{eq} \cdot \nabla q\, dx = \int_K \underline{\boldsymbol{\psi}}_h \cdot \nabla q\, dx = \int_K \underline{\boldsymbol{\psi}}_h \cdot \underline{\mathbf{p}}\, dx.$$

Combining the last two equations we obtain (6.9)

The following theorem is the main result and shows that $\underline{\underline{\mathbf{p}}}_h^{eq}$ is an equilibrated moment tensor and thus fulfills all requirements of the two-energies principle.

THEOREM 6.6. *Let* $k \geq 2$. *If the moment tensor* $\underline{\underline{\mathbf{p}}}_h^{eq}$ *and the auxiliary vector field* $\underline{\boldsymbol{\psi}}_h^{eq}$ *are constructed by* (6.8) *and* (6.5), *respectively, then* $\underline{\underline{\mathbf{p}}}_h^{eq} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega)$ *is equilibrated, i.e.,*

$$\nabla \cdot \nabla \cdot \underline{\underline{\mathbf{p}}}_h^{eq} = f_h \quad \textit{in } L^2(\Omega).$$

Proof. It follows from (6.8a) that the normal components of $\underline{\underline{\mathbf{p}}}_h^{eq}$ are continuous on edges. Hence, $\underline{\underline{\mathbf{p}}}_h^{eq} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega)$ Let $K \in \mathcal{T}_h(\Omega)$. From Remark 6.5 we know that the compatibility condition (6.9) is satisfied. We apply partial integration and insert the rules (6.8a), (6.8b) for the construction of $\underline{\underline{\mathbf{p}}}_h^{eq}$ to obtain

$$\begin{aligned}
\int_K \boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{p}}}_h^{eq} \cdot \underline{\mathbf{q}}\, dx &= -\int_K \underline{\underline{\mathbf{p}}}_h^{eq} : \boldsymbol{\nabla}\underline{\mathbf{q}}\, dx + \int_{\partial K} \underline{\underline{\mathbf{p}}}_h^{eq}\, \mathbf{n}_{\partial K} \cdot \underline{\mathbf{q}}\, ds \\
&= -\Big(-\int_K \underline{\boldsymbol{\psi}}_h^{eq} \cdot \underline{\mathbf{q}}\, dx + \int_{\partial K} \underline{\underline{\widehat{\mathbf{p}}}}\, \mathbf{n}_{\partial K} \cdot \underline{\mathbf{q}}\, ds \Big) + \int_{\partial K} \underline{\underline{\widehat{\mathbf{p}}}}\, \mathbf{n}_{\partial K} \cdot \underline{\mathbf{q}}\, ds \\
&= \int_K \underline{\boldsymbol{\psi}}_h^{eq} \cdot \underline{\mathbf{q}}\, dx\ , \qquad \underline{\mathbf{q}} \in P_{\ell-1}(K)^2.
\end{aligned} \tag{6.10}$$

Since both $\boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{p}}}_h^{eq}$ and $\underline{\boldsymbol{\psi}}_h^{eq}$ live in $P_{\ell-1}(K)^2$, it follows from (6.10) that

$$\boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{p}}}_h^{eq} = \underline{\boldsymbol{\psi}}_h^{eq}. \tag{6.11}$$

The left-hand side is contained in $\mathbf{H}(\mathrm{div}, \Omega)$ since it holds for the right-hand side due to Lemma 6.2. Moreover it follows that $\underline{\underline{\mathbf{p}}}_h^{eq} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega)$ and

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{p}}}_h^{eq} = \boldsymbol{\nabla} \cdot \underline{\boldsymbol{\psi}}_h^{eq} = f_h$$

and the proof is complete. □

Usually mixed methods for the treatment of the Hellan–Herrrmann–Johnson formulation use finite elements for the moment tensors that are $H(\mathrm{div}^2)$ nonconforming. This is due to the fact that no simple conforming elements are known. The reader will have observed that the equilibrated moment tensors are constructed in $\underline{\underline{\mathbf{M}}}_h \cap \underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega)$. Thus we have implicitly an $\underline{\underline{\mathbf{H}}}(\mathrm{div}^2)$-conforming finite element space. We conclude from the efficiency considerations in the next section that this finite element (sub)space is sufficiently large.

REMARK 6.7. *We note that the divergence of a tensor was defined row-wise in* (3.4). *If we had chosen a column-wise definition, then we would have obtained the transposed tensor* $\underline{\underline{\mathbf{p}}}_h^{eq,T}$ *of the result* (6.8). *It follows that also* $\underline{\underline{\mathbf{p}}}_h^{eq,T} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega)$ *and* $\mathrm{div}\,\mathrm{div}\,\underline{\underline{\mathbf{p}}}_h^{eq,T} = f_h$. *Therefore we may use also the symmetrical part*

$$\underline{\underline{\mathbf{p}}}_h^{eq,sym} = \frac{1}{2}\big(\underline{\underline{\mathbf{p}}}_h^{eq} + \underline{\underline{\mathbf{p}}}_h^{eq,T}\big)$$

for computing the term (5.3a) *of the error bound, i.e.,*

$$\eta_{K,1}^{eq,s} := \|D^2 u_h - \underline{\underline{\mathbf{p}}}_h^{eq,sym}\|_{0,K}, \quad K \in \mathcal{T}_h(\Omega). \tag{6.12}$$

Since the symmetrical part and the antisymmetrical part of a tensor are L^2-orthogonal, it follows that

$$\eta_{K,1}^{eq,s} \le \eta_{K,1}^{eq}, \quad K \in \mathcal{T}_h(\Omega). \tag{6.13}$$

Indeed, numerical results below show that the error bound can be reduced by about 30% *in this way.*

7. Efficiency of the equilibrated error estimator. A residual-type a posteriori error estimator has been derived and analyzed in [28] for the IPDG approximation of the biharmonic problem. It is based on the Ciarlet–Raviart mixed formulation, and its adaptation to the Hellan–Hermann–Johnson based IPDG approximation (4.4) reads as follows:

$$(\eta_h^{res})^2 = \sum_{K \in \mathcal{T}_h(\Omega)} (\eta_K^{res})^2 + \sum_{E \in \mathcal{E}_h(\bar{\Omega})} \Big((\eta_{E,1}^{res})^2 + (\eta_{E,2}^{res})^2 + (\eta_{E,c}^{res})^2\Big), \tag{7.1}$$

where the element residual η_K^{res} and the edge residuals $\eta_{E,1}^{res}, \eta_{E,2}^{res}, \eta_{E,c}^{res}$ are given by

$$\begin{aligned}
(\eta_K^{res})^2 &:= h_K^4\ \|f - \Delta^2 u_h\|_{0,K}^2, \\
(\eta_{E,1}^{res})^2 &:= h_E^3\ \|\mathbf{n}_E \cdot [\boldsymbol{\nabla}\Delta u_h\|_{0,E}^2, \\
(\eta_{E,2}^{res})^2 &:= h_E\ \Big(\|\mathbf{n}_E \cdot [D^2 u_h]_E\ \mathbf{n}_E\|_{0,E}^2 + \|\mathbf{t}_E \cdot [D^2 u_h]_E\ \mathbf{n}_E\|_{0,E}^2\Big), \\
(\eta_{E,c}^{res})^2 &:= h_E^{-1}\ \|\mathbf{n}_E \cdot [\boldsymbol{\nabla} u_h]_E\|_{0,E}^2 + h_E^{-3}\ \|[u_h]_E\|_{0,E}^2.
\end{aligned} \tag{7.2}$$

A slight generalization of the efficiency estimate from [28] shows

$$(\eta_h^{res})^2 \lesssim \|u - u_h\|_{2,h,\Omega}^2 + osc_h^2(f). \tag{7.3}$$

The efficiency of the equilibrated a posteriori error estimator η_h^{eq} follows from (7.3) and the following result.

LEMMA 7.1. *Let* $\eta_K^{eq}, K \in \mathcal{T}_h(\Omega)$, *and* $osc_h(f)$ *be given by* (5.3a) *and* (5.6), *and let* η_h^{res} *be the residual-type a posteriori error estimator* (7.1). *Then there holds*

$$\sum_{K \in \mathcal{T}_h(\Omega)} (\eta_{K,1}^{eq})^2 \lesssim (\eta_h^{res})^2 + osc_h^2(f). \tag{7.4}$$

Proof. Let $K \in \mathcal{T}_h(\Omega)$ and $E \in \mathcal{E}_h(\partial K)$. Due to (6.8a) and (4.10c) we have $\underline{\underline{\mathbf{p}}}_h^{eq} = \widehat{\underline{\underline{\mathbf{p}}}} = \{D^2 u_h\}_E - \frac{\alpha_1}{h_E}\ [\mathbf{n}_E \cdot \boldsymbol{\nabla} u_h^T]_E$. Hence,

$$\begin{aligned}
\mathbf{n}_E \cdot (\underline{\underline{\mathbf{p}}}_h^{eq} - D^2 u_h)\mathbf{n}_E &= \mathbf{n}_E \cdot \big[\{D^2 u_h\}_E - \cdot D^2 u_h\big]\mathbf{n}_E - \frac{\alpha_1}{h_E}\ [\mathbf{n}_E \cdot \boldsymbol{\nabla} u_h]_E, \\
\mathbf{t}_E \cdot (\underline{\underline{\mathbf{p}}}_h^{eq} - D^2 u_h)\mathbf{n}_E &= \mathbf{t}_E\ \cdot \big[\{D^2 u_h\}_E - D^2 u_h\big]\mathbf{n}_E.
\end{aligned}$$

It follows that

$$|\mathbf{n}_E \cdot (\underline{\underline{\mathbf{p}}}_h^{eq} - D^2 u_h)\mathbf{n}_E| \le \frac{1}{2}\ |\mathbf{n}_E \cdot [D^2 u_h]_E\ \mathbf{n}_E| + \frac{\alpha_1}{h_E}\ |\mathbf{n}_E \cdot [\boldsymbol{\nabla} u_h]_E|, \tag{7.5a}$$

$$|\mathbf{t}_E \cdot (\underline{\underline{\mathbf{p}}}_h^{eq} - D^2 u_h)\mathbf{n}_E| \le \frac{1}{2}\ |\mathbf{t}_E \cdot [D^2 u_h]_E\ \mathbf{n}_E|. \tag{7.5b}$$

Moreover, in view of (6.11) and (6.8c) we have

$$\boldsymbol{\nabla} \cdot (\underline{\underline{\mathbf{p}}}_h^{eq} - D^2 u_h) = \underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h, \tag{7.6a}$$

$$\int\limits_K (\underline{\mathbf{p}}_{h,eq}^{(i)} - \underline{\mathbf{z}}_h^{(i)}) \cdot \mathbf{curl}(b_K q)\, dx = 0, \quad q \in P_{\ell-2}(K),\ 1 \le i \le 2. \tag{7.6b}$$

Observing (7.5) and (7.6) we apply Lemma 6.4 to $\underline{\underline{\mathbf{p}}}_h^{eq} - D^2 u_h \in P_k^{2\times 2}$, recall (7.2) and obtain

$$\begin{aligned}
\|\underline{\underline{\mathbf{p}}}_h^{eq} - D^2 u_h\|_{0,K}^2 &\lesssim h_K^2\ \|\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h(\partial K)} \frac{\alpha_1^2}{h_E}\ \|\mathbf{n}_E \cdot [\boldsymbol{\nabla} u_h]_E\|_{0,E}^2 \\
&+ \sum_{E \in \mathcal{E}_h(\partial K)} h_E\ \Big(\|\mathbf{n}_E \cdot [D^2 u_h]_E\ \mathbf{n}_E\|_{0,E}^2 + \|\mathbf{t}_E \cdot [D^2 u_h]_E\ \mathbf{n}_E\|_{0,E}^2\Big) \\
&\le h_K^2\ \|\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h(\partial K)} \Big((\eta_{E,c}^{res})^2 + (\eta_{E,2}^{res})^2\Big).
\end{aligned} \tag{7.7}$$

Now we turn to the estimation of $\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h$. We have for $E \in \mathcal{E}_h(\partial K)$ in view of (6.5a) and (4.10d) $\underline{\boldsymbol{\psi}}_h^{eq} = Q_{\ell-1}^E \widehat{\underline{\boldsymbol{\psi}}} = \{D^2 u_h\}_E + \frac{\alpha_2}{h_E^3} Q_{\ell-1}^E([u_h]_E)$ and

$$\mathbf{n}_E \cdot (\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h) = \mathbf{n}_E \cdot \big(\{\boldsymbol{\nabla} \cdot D^2 u_h\}_E - \boldsymbol{\nabla} \cdot D^2 u_h\big) + \frac{\alpha_2}{h_E^3} Q_{\ell-1}^E([u_h]_E).$$

Noting that $\boldsymbol{\nabla} \cdot D^2 u_h = \boldsymbol{\nabla} \Delta u_h$ we obtain

$$|\mathbf{n}_E \cdot (\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h)| \le \frac{1}{2}\ |\mathbf{n}_E \cdot [\boldsymbol{\nabla} \Delta u_h]_E| + \frac{\alpha_2}{h_E^3} |Q_{\ell-1}^E([u_h]_E)|. \tag{7.8}$$

Moreover, taking (6.1) and (6.5c) into account, it holds

$$\boldsymbol{\nabla} \cdot (\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h) = f_h - \Delta^2 u_h \quad \text{in } K, \tag{7.9a}$$

$$\int\limits_K (\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h) \cdot \mathbf{curl}(b_K q)\, dx = 0, \quad q \in P_{\ell-3}(K). \tag{7.9b}$$

Due to (7.8) and (7.9a),(7.9b), an application of Lemma 6.1 to $\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h \in P_{\ell-1}(K)^2$ yields

$$\|\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h\|_{0,K}^2 \lesssim h_K^2 \ \|f_h - \Delta^2 u_h\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h(\partial K)} h_E \ \|\mathbf{n}_E \cdot [\boldsymbol{\nabla} \Delta u_h]_E\|_{0,E}^2 + \sum_{E \in \mathcal{E}_h(\partial K)} \frac{\alpha_2^2}{h_E^3} \|[u_h]_E\|_{0,E}^2. \tag{7.10}$$

Using the local quasi-uniformity once more, we have $h_E \sim h_K$ for $E \in \mathcal{E}_h(\partial K)$ and estimate the bounds above in terms of the residual estimators (7.2)

$$h_K^2 \|\underline{\boldsymbol{\psi}}_h^{eq} - \boldsymbol{\nabla} \cdot D^2 u_h\|_{0,K}^2 \lesssim (\eta_{E,2}^{res})^2 + h_K^4 \|f - f_h\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h(\partial K)} \Big((\eta_{E,2}^{res})^2 + (\eta_{E,3}^{res})^2 \Big).$$

We insert this bound into (7.7), sum over all $K \in \mathcal{T}_h(\Omega)$, and the proof is complete. □

THEOREM 7.2. *Let $u \in H_0^2(\Omega)$ be the solution of the biharmonic problem* (3.1)*, and let $u_h \in V_h$ be the IPDG approximation. Moreover, let $\eta_{K,i}^{eq}, \eta_{E,i}^{eq}, 1 \le i \le 2$, and $osc_h(f)$ be given by* (5.3a)*–*(5.3c) *and* (5.6)*. Then there exists a constant $C > 0$ depending on the polynomial degree k, the local geometry of the triangulation, and on the penalty parameters $\alpha_i, 1 \le i \le 2$, such that*

$$\begin{aligned} &\sum_{K \in \mathcal{T}_h(\Omega)} \Big((\eta_{K,1}^{eq})^2 + (\eta_{K,2}^{eq})^2 \Big) + \sum_{E \in \mathcal{E}_h(\bar{\Omega})} ((\eta_{E,1}^{eq})^2 + (\eta_{E,2}^{eq})^2) \\ &\le C \ \Big(\|u - u_h\|_{2,h,\Omega}^2 + osc_h^2(f) \Big). \end{aligned} \tag{7.11}$$

Proof. The assertion follows directly from (5.12), (7.3), and (7.4). □

Since the residual a posteriori error estimator is known to be efficient [28], the error bounds from the two-energies principle are also efficient.

8. Numerical results. We provide a detailed documentation of the performance of the adaptive IPDG method for an illustrative example taken from [29] which has also been used in [14].

EXAMPLE 8.1. We choose Ω as the L-shaped domain $\Omega := (-1,+1)^2 \setminus ([0,1) \times (-1,0])$ and choose f in (3.1a) such that

$$u(r,\varphi) = \Big(r^2 \cos^2 \varphi - 1\Big)^2 \Big(r^2 \sin^2 \varphi - 1\Big)^2 \; r^{1+z} \; g(\varphi) \tag{8.1}$$

is the exact solution of the biharmonic boundary-value problem (3.1), where

$$\begin{aligned} &g(\varphi) := \\ &\Big(\tfrac{1}{z-1} \sin \tfrac{3(z-1)\pi}{2} - \tfrac{1}{z+1} \sin \tfrac{3(z+1)\pi}{2}\Big)\Big(\cos((z-1)\varphi) - \cos((z+1)\varphi)\Big) - \\ &\Big(\tfrac{1}{z-1} \sin((z-1)\varphi) - \tfrac{1}{z+1} \sin((z+1)\varphi)\Big)\Big(\cos \tfrac{3(z-1)\pi}{2} - \cos \tfrac{3(z-1)\pi}{2}\Big), \end{aligned}$$

and $z \approx 0.54448$ is a non-characteristic root of $\sin^2(\frac{3z\pi}{2}) = z^2 \sin^2(\frac{3\pi}{2})$.

The penalty parameters have been chosen as $\alpha_1 := 12.5\,(k+1)^2$ and $\alpha_2 := 2.5\,(k+1)^6$.

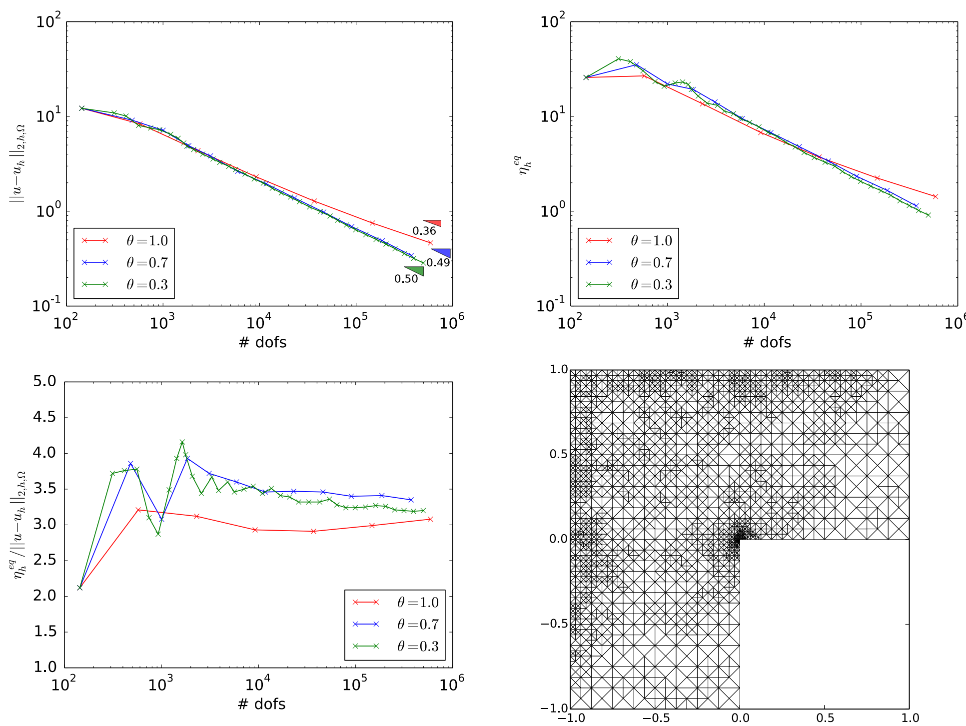

FIG. 8.1. *Error, estimator, effectivity index, and adaptively generated mesh ($k = 2$).*

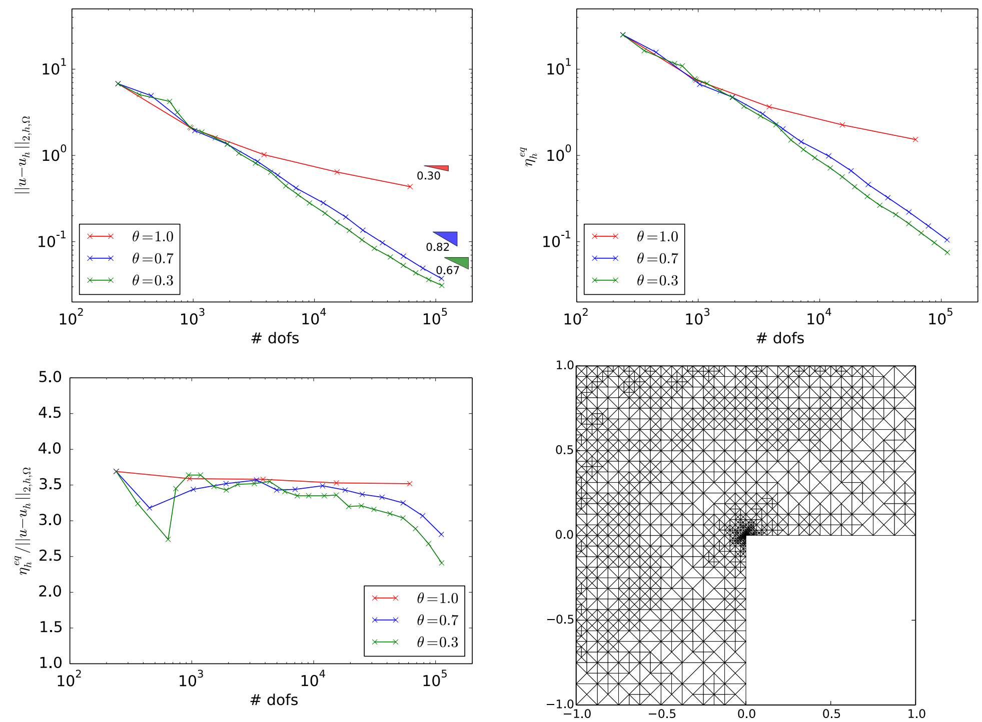

FIG. 8.2. *Error, estimator, effectivity index, and adaptively generated mesh ($k = 3$).*

We make use of the notation

$$\eta^{eq}_{\mathcal{E}_h,\alpha} := \Big(\sum_{E \in \mathcal{E}_h(\bar\Omega)} \big(\alpha_1 (\eta^{eq}_{E,1})^2 + \alpha_2 (\eta^{eq}_{E,2})^2\big) \Big)^{1/2}, \tag{8.2a}$$

$$\eta^{eq}_h := \Big(\sum_{K \in \mathcal{T}_h(\Omega)} (\eta^{eq}_{K,1})^2 \Big)^{1/2} + \eta^{eq}_{\mathcal{E}_h,\alpha}\,, \tag{8.2b}$$

$$\eta^{eq,s}_h := \Big(\sum_{K \in \mathcal{T}_h(\Omega)} (\eta^{eq,s}_{K,1})^2 \Big)^{1/2} + \eta^{eq}_{\mathcal{E}_h,\alpha}\,, \tag{8.2c}$$

where $\eta^{eq,s}_{K,1}$ has been defined in (6.12). Note that the re-definition of η^{eq}_h in (8.2b) differs from (5.7) in so far as we have omitted the second term of the right-hand side in (5.7) because according to (5.12) it can be estimated from above by the third term. For polynomial degree $2 \le k \le 5$ and bulk parameters $\theta = 1.0$ (uniform refinement), $\theta = 0.7$, and $\theta = 0.3$ Figures 8.1–8.4 display

- the global discretization error $u - u_h$ in the mesh-dependent IPDG-norm $\|\cdot\|_{2,h,\Omega}$ (top left) and the error estimator η^{eq}_h (top right) as a function of the total number of degrees of freedom (dofs) on a logarithmic scale,
- the associated effectivity index $\eta^{eq}_h/\|u - u_h\|_{2,h,\Omega}$ (bottom left),
- the adaptively generated mesh ($\theta = 0.7$) at refinement level 7 for $k = 2$, level 9 for $k = 3$, level 11 for $k = 4$, and level 13 for $k = 5$ (bottom right).

We observe a significant refinement in a vicinity of the reentrant corner where the solution has a singularity and some refinement in regions near the upper and left

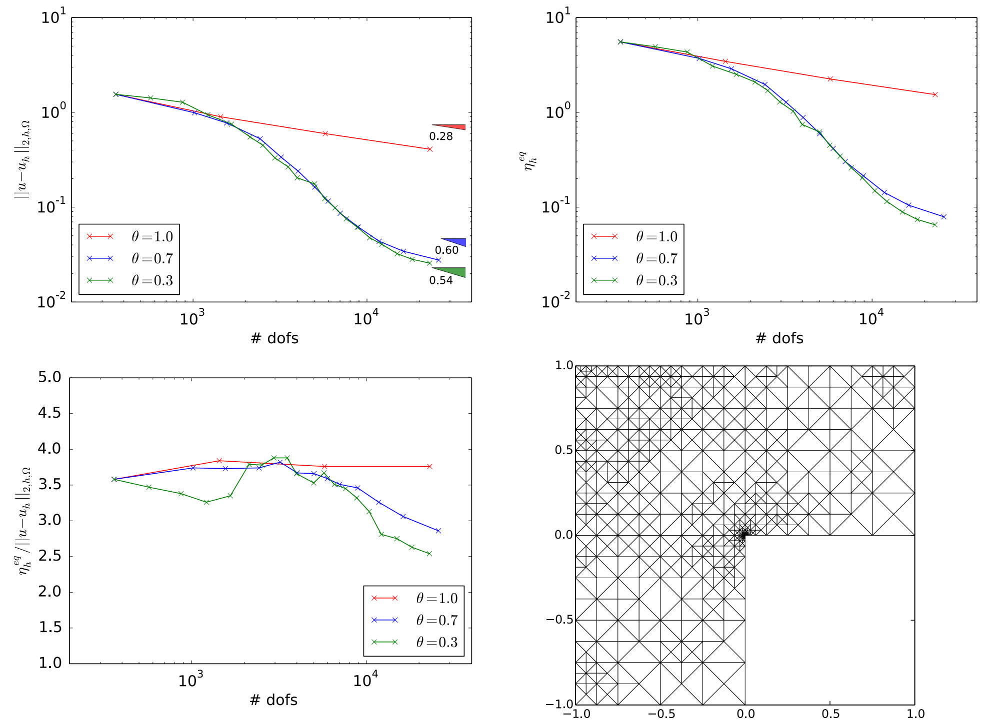

FIG. 8.3. *Error, estimator, effectivity index, and adaptively generated mesh ($k = 4$).*

boundary segments of the computational domain where second derivatives of the solution have local peaks. As expected, the refinement is less pronounced for higher polynomial degree k. Moreover, for $k = 2$ the beneficial effect of adaptive refinement sets in for a total number of DOFs (# DOFs) exceeding 10^4, whereas for $3 \leq k \leq 5$ it occurs for # DOFs $\approx 10^3$ and is much more pronounced than for $k = 2$. The effectivity index is between 2.0 and 4.5 for all polynomial degrees $2 \leq k \leq 5$.

We note that the computation of the equilibrated moment tensor is ill-conditioned. The condition number deteriorates significantly with decreasing mesh size and increasing polynomial degree k. For $k = 4$ and $k = 5$, Figures 8.3 and 8.4 only display the results up to refinement levels before roundoff errors have an influence on the numerical results.

Table 8.1 lists results of the computation for $k = 3$ and $\theta = 0.3$ and addresses certain components of the error estimator η_h^{eq}. By using the symmetrical part $\eta_h^{eq,s}$ (cf. (8.2c)) as suggested in Remark 6.7, the error bounds and therefore also the associated effectivity indices $\eta_h^{eq,s}/\|u - u_h\|_{2,h,\Omega}$ can be reduced by 15 to 20 %. The weighted edge-related terms as given by $\eta_{\mathcal{E}_h,\alpha}^{eq}$ contribute only about $12 - 15\,\%$ to the overall error estimator.

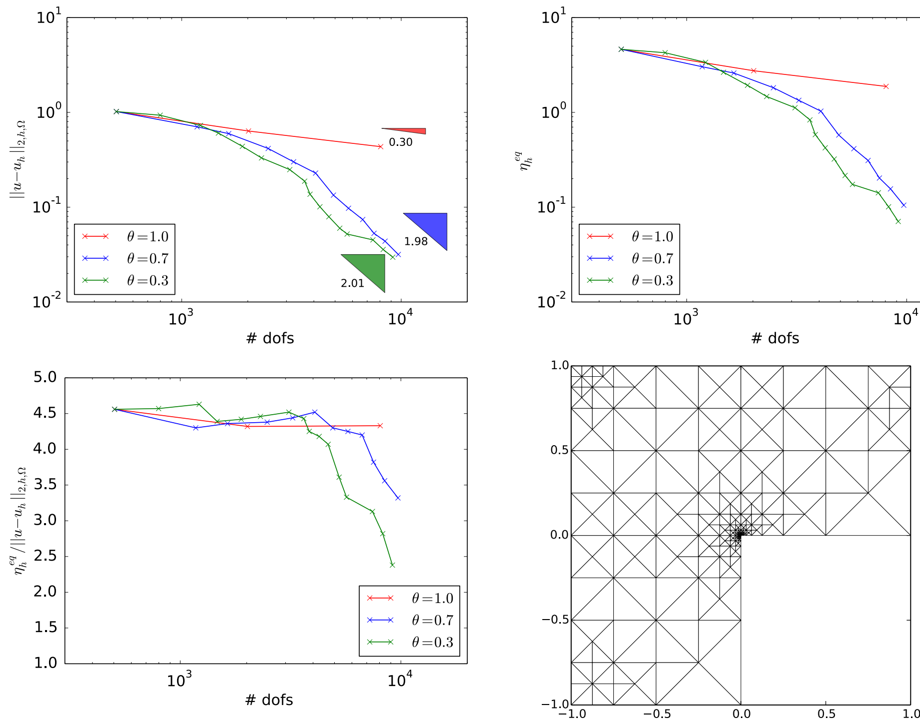

Fig. 8.4. *Error, estimator, effectivity index, and adaptively generated mesh ($k = 5$).*

Table 8.1
Results for $k = 3$ and $\theta = 0.3$

level	# dofs	$\|u-u_h\|_{2,h,\Omega}$	η_h^{eq}	$\eta_h^{eq,s}$	$\eta_{\mathcal{E}_h,\alpha}^{eq}$	effectivity
0	240	$6.78 \cdot 10^0$	$2.50 \cdot 10^1$	$2.04 \cdot 10^1$	$3.06 \cdot 10^0$	3.69
2	640	$4.23 \cdot 10^0$	$1.16 \cdot 10^1$	$9.85 \cdot 10^0$	$1.77 \cdot 10^0$	2.74
4	940	$2.13 \cdot 10^0$	$7.75 \cdot 10^0$	$6.41 \cdot 10^0$	$1.19 \cdot 10^0$	3.64
6	1520	$1.60 \cdot 10^0$	$5.56 \cdot 10^0$	$4.61 \cdot 10^0$	$7.93 \cdot 10^{-1}$	3.48
8	2380	$1.06 \cdot 10^0$	$3.71 \cdot 10^0$	$3.06 \cdot 10^0$	$4.77 \cdot 10^{-1}$	3.51
10	4360	$6.39 \cdot 10^{-1}$	$2.27 \cdot 10^0$	$1.86 \cdot 10^0$	$2.84 \cdot 10^{-1}$	3.55
12	7340	$3.49 \cdot 10^{-1}$	$1.17 \cdot 10^0$	$9.67 \cdot 10^{-1}$	$1.47 \cdot 10^{-1}$	3.35
14	12210	$2.14 \cdot 10^{-1}$	$7.16 \cdot 10^{-1}$	$5.89 \cdot 10^{-1}$	$8.83 \cdot 10^{-2}$	3.35
16	19380	$1.35 \cdot 10^{-1}$	$4.34 \cdot 10^{-1}$	$3.57 \cdot 10^{-1}$	$5.43 \cdot 10^{-2}$	3.20
18	31190	$8.37 \cdot 10^{-2}$	$2.64 \cdot 10^{-1}$	$2.18 \cdot 10^{-1}$	$3.23 \cdot 10^{-2}$	3.16
20	54040	$5.31 \cdot 10^{-2}$	$1.62 \cdot 10^{-1}$	$1.33 \cdot 10^{-1}$	$1.96 \cdot 10^{-2}$	3.04

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