Preliminary Examination Syllabus: Real Analysis

University of Houston, Department of Mathematics

There are many good books covering these topics; the following list is not comprehensive.

(1) S. Axler, "Measure, Integration & Real Analysis", 2020 (free from author's website)
(2) R. Bass, "Real Analysis for Graduate Students", 2014 (free from author's website)
(3) G. Folland, "Real Analysis: Modern Techniques and their Applications", 1999
(4) W. Rudin, "Real and Complex Analysis", 1987
(5) D. Salamon, "Measure and Integration", 2016
(6) J. Yeh, "Theory of Measure and Integration", 2006

Background: Familiarity is expected with standard undergraduate topics, whether or not they are covered in Math 6320-6321, such as linear algebra, complex numbers, countable sets and cardinality, equivalence relations, and basic topology in metric spaces. This includes precise definitions of limits, continuity, uniform convergence, open, closed, complete, compact, product space, etc., as well as statements of important results such as Heine–Borel, compactness of products of compact metric spaces, Stone–Weierstrass, etc.

1. Measures and Integration

(1) Algebra, σ-algebra, measurable space, measurable set. Borel σ-algebra.

(2) Measures: σ-finite, finite, probability. Completion of a measure.

(3) Carathéodory extension theorem: premeasure on an algebra, construction of an outer measure μ^*, the σ-algebra of μ^*-measurable sets.

(4) Lebesgue and Lebesgue–Stieltjes measures on $\mathbb{R}$. Null sets including Cantor set.

(5) Measures on metric spaces: regularity, Borel, Radon.

(6) Monotone class theorem. Product σ-algebra, product measure, Lebesgue measure on $\mathbb{R}^n$.

(7) Signed measure, positive and negative sets. Hahn and Jordan decompositions. Total variation measure. Complex measures.

(8) Mutually singular measures, absolutely continuous measures, Cantor–Lebesgue function.

(9) Measurable functions. Approximation by simple functions.

(10) Integration in measure spaces: non-negative measurable functions, integrable functions.

(11) Measurable transformations: pushforward measures and integration.

(12) The Lebesgue integral on $\mathbb{R}^n$ and its properties: translation invariance, change of variables, etc. Characterization of Riemann integrability and equality with the Lebesgue integral.

(13) Convergence theorems: Monotone, Fatou, dominated convergence, etc.

(14) Almost everywhere equality, convergence. Other types of convergence: in measure, in L^p, uniform, in L^∞, etc, and their relationships.

(15) Egorov's theorem, Luzin's theorem, Fubini–Tonelli theorem.

2. Differentiation

(1) Radon–Nikodym theorem, Lebesgue decomposition theorem.

(2) Differentiation of measures on $\mathbb{R}^n$, Lebesgue differentiation theorem, Vitali covering lemma, Lebesgue density theorem.

(3) Bounded variation functions on $\mathbb{R}$, Lipschitz continuity, absolutely continuous functions. BV functions are the difference of increasing functions, hence a.e.-differentiable.

(4) Fundamental theorems of calculus for AC functions.

(5) Connection between absolute continuity of a function F and of the Lebesgue–Stieltjes measure m_F; equality between a.e. derivative of F and Radon–Nikodym derivative of m_F. *(An important motivation here involves probability theory via pushforward measures on* $\mathbb{R}$*, but this application is not on the exam.)*

(6) Characterization of convex functions via absolute continuity; Jensen's inequality.

3. Basics of functional and Fourier analysis

(1) L^p spaces: conjugate exponents, basic inequalities (Chebyshev, Hölder, Minkoswki). Density of simple functions. Convolution.

(2) Normed vector spaces, dual space, duals of L^p spaces. Hahn–Banach theorem and its consequences, isometric embedding $X \to X^{**}$.

(3) Measures as linear functionals: Riesz–Markov–Kakutani representation theorem.

(4) Banach spaces: L^p, ℓ^p, C, C_0, $L(X, Y)$. Separability. Baire category theorem and its consequences, uniform boundedness principle (Banach–Steinhaus theorem), open mapping theorem, closed graph theorem.

(5) Hilbert spaces: inner product, Cauchy–Schwarz, parallelogram law, polarization identity, orthogonal complements and projections, Riesz representation (relate H and H^*), orthonormal sets, Gram–Schmidt procedure, Bessel's inequality, completeness, Parseval's identity, orthonormal basis, uniqueness of the Hilbert space of given dimension.

(6) Fourier transform on $L^1(\mathbb{R}^n)$ and its basic properties such as the Riemann–Lebesgue lemma, and some form of the Fourier inversion formula. Fourier series using Hilbert spaces (L^2-convergence of Fourier series of an L^2 function).