ON HOLOMORPHIC SECTIONAL CURVATURE AND FIBRATIONS

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By

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Abstract

In this dissertation, we prove the existence of a metric of definite holomorphic sectional curvature on certain compact fibrations. The basic idea for these curvature computations is to use the already available information on the signs of the holomorphic sectional curvatures along the base and the fibers of the fibration, and construct an appropriate warped metric on the total space. For a few specific fibrations, like Hirzebruch surfaces, isotrivial families of curves, and product manifolds, we shall also comment on the pinching constants of the holomorphic sectional curvatures. All these results are either in the case of strictly positive holomorphic sectional curvature, or in the case of strictly negative holomorphic sectional curvature. At the end of this dissertation, we give a few examples to show that the sign of the holomorphic sectional curvature of a fibration might not be what we would expect in the cases where the base or the fibers have semi-definite holomorphic sectional curvatures.

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Chapter 1

Introduction

The simplest example of a fibration in our context is a trivial fiber bundle $f : X \times Y \to Y$, where X and Y are complex manifolds and the mapping f is the projection onto the second coordinate. In general however, a fibration does not even possess a local product structure. Nevertheless, we can always talk about the two "directions" of a fibration: base and fibers. Therefore, in order to address a problem on the holomorphic sectional curvature of a fibration, i.e., sectional curvature along holomorphic tangent directions, it might be helpful to consider the behavior of the holomorphic sectional curvature along (i) the fibers, (ii) the base, and (iii) the skew directions. If the holomorphic sectional curvatures along the base and the fibers of a fibration have same signs, then in this dissertation, we would like to find a Hermitian metric on the total space with the same sign of the holomorphic sectional curvature as on the base and the fibers.

For the product of two complex manifolds, the product metric is the required metric, as we shall see in Theorem 3.2 and Theorem 4.2. However, for a more general fibration, we need to consider the more sophisticated concept of a *warped product* *metric*: $g + \lambda h$, where g and h are Hermitian metrics obtained from the base and the fiber directions, respectively, and λ is a constant, called *warp factor*. The idea of a warped product metric was first promoted by Bishop-O'Neill in [BO69] to obtain metrics of negative curvature. This concept is further seen to be used by Hitchin in [Hit75] in order to produce a metric of positive holomorphic sectional curvature on the Hirzebruch surfaces $\mathbb{F}_n = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}_1}(n) \oplus \mathcal{O}_{\mathbb{P}_1}\big),\ n \in \{0, 1, \ldots\}$. Cheung also used a warp factor λ in [Che89] to prove the negative holomorphic sectional curvature of a compact fibration when the base and the fibers of the fibration carry a metric of negative holomorphic sectional curvature.

In this dissertation, we shall extend the work of Hitchin and Cheung, and prove some new results on the holomorphic sectional curvature of a fibration. This dissertation is organized as follows:

In Chapter 2, we shall recall some elementary definitions which will be required in the remaining chapters of this dissertation.

Chapter 3 is focused on the Hirzebruch surfaces. The main purpose of this chapter is to evaluate a pinching constant for the positive holomorphic sectional curvature of Hitchin's metric on the Hirzebruch surfaces. We also discuss the pinching of a product manifold $M \times N$ when both M and N carry a Hermitian metric of positive holomorphic sectional curvature, in order to address the case of the 0-th Hirzebruch surface. The contents of this chapter have already been published in [ACH15].

Chapter 4 is on the holomorphic sectional curvature of an isotrivial family of curves over a smooth curve, $f : \mathcal{F} \to C$, where the base C, and the fiber F are curves of genus greater than or equal to 2. Both C and F carry Hermitian metrics (Poincaré metric) of constant negative holomorphic sectional curvature equal to -1. The main result of this chapter (Theorem 4.1) shows that, in this case, there exists a Hermitian metric of $-\frac{1}{2}$-pinched holomorphic sectional curvature on $\mathcal{F}$. Theorem 4.1 on isotrivial families of curves is the negative curvature analog of Theorem 3.1 on the Hirzebruch surfaces. However, surprisingly the pinching constant $-\frac{1}{2}$ in Theorem 4.1 is independent of how "twisted" the family is. We shall also prove the (semi-) negative holomorphic sectional curvature analog of Theorem 3.2, and an equivalence between the negative holomorphic sectional curvatures of a complex manifold and its covering space. These two theorems, in the 1-dimensional case, are crucial in the proof of Theorem 4.1. However, we observe that these two theorems are valid even for higher dimensional complex manifolds.

In Chapter 5, we show the existence of a Hermitian metric of positive holomorphic sectional curvature on a compact fibration when the base and the fibers of the fibration, both possess metrics of positive holomorphic sectional curvature. This result could perhaps be considered to be the main theorem of this dissertation. It is widely believed that the "curvature decreasing property for subbundles" is very helpful in proving results for negative curvature, but usually not as useful in positive curvature. As we shall see in this chapter, the same "curvature decreasing" property helps us prove the existence of a metric of positive holomorphic sectional curvature on a compact fibration. The results and proofs of this chapter generally follow the results and proofs in [Che89], but differ in certain key aspects.

In Chapter 6, we give a few examples to show that the results proved in Chapter 5 and [Che89] for definite holomorphic sectional curvature do not extend to the case of semi-definite holomorphic sectional curvature. These examples show that the warped product metric might not be helpful to obtain semi-definite holomorphic sectional curvature on the total space, even when the warp factor is not constant. However, it is not known whether there exist other metrics for which we can extend the results of Chapter 5 to semi-definite cases.

Chapter 2

Definitions

2.1 Fibrations

In this dissertation, we will call a holomorphic map $\pi : M \to N$ between two complex manifolds M and N a *fibration* if π has maximal rank everywhere, i.e., is a submersion, and is surjective. We refer to a fibration with compact domain M as a *compact fibration*.

Loosely speaking, fibrations are generalizations of fiber bundles. Unlike a fiber bundle, the fibers of a fibration need not be isomorphic to each other.

2.2 Hermitian metric

Let M be an m-dimensional manifold with local coordinates $(z_1, \ldots, z_m)$. A *Hermitian metric* $g : TM \times TM \to \mathbb{C}$ on M is a Hermitian inner product on the tangent space T_pM at each point $p \in M$, varying smoothly with respect to the points in M. In a neighborhood U of a point $p \in M$, the metric g may be locally represented as

$$g = \sum_{i,j=1}^{m} g_{i\bar{j}} dz_i \otimes d\bar{z}_j,$$

where $(g_{i\bar{j}})$ is a Hermitian matrix with $g_{ij} \in C^\infty(U, \mathbb{C})$ for all $i, j = 1, \ldots, m$. Under the usual abuse of terminology, we will alternatively refer to the associated (1,1)-form $\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{m} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ as the metric on M. The metric is called *Kähler* if ω is d-closed. It is called *Hodge* if it is Kähler and the cohomology class of ω is rational.

2.3 Components of curvature tensor

The components $R_{i\bar{j}k\bar{l}}$ of the curvature tensor R associated with the metric connection are locally given by the formula

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{m} g^{p\bar{q}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{j}}}{\partial \bar{z}_l}, \tag{2.1}$$

for $i, j, k, l = 1, \ldots, m$.

2.4 Holomorphic sectional curvature

If $\xi = \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial z_i}$ is a non-zero complex tangent vector at $p \in M$, then the *holomorphic sectional curvature* $K(\xi)(p)$ is given by

$$K(\xi)(p) = \left(2 \sum_{i,j,k,l=1}^{m} R_{i\bar{j}k\bar{l}}(p) \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l \right) / \left(\sum_{i,j,k,l=1}^{m} g_{i\bar{j}}(p) g_{k\bar{l}}(p) \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l \right). \tag{2.2}$$

Note that the holomorphic sectional curvature of ξ is clearly invariant under multiplication of ξ with a real non-zero scalar, and it thus suffices to consider unit tangent vectors, for which the value of the denominator is 1.

It is a basic fact that the holomorphic sectional curvature of a Kähler metric completely determines the curvature tensor $R_{i\bar{j}k\bar{l}}$ ([KN96, Proposition 7.1, p. 166]).

We shall use the abbreviation HSC for "holomorphic sectional curvature" in the remaining part of the dissertation.

2.5 Pinching

2.5.1 Positive HSC

For a constant $c^+ \in (0, 1]$, we say that the positive HSC is c^+-*pinched* if

$$(1 \geq)\ \frac{\inf_\xi K(\xi)}{\sup_\xi K(\xi)} = c^+,$$

where the infimum and supremum are taken over all non-zero (or unit) tangent vectors across the entire manifold.

2.5.2 Negative HSC

For a constant $c^- \in [-1, 0)$, we say that the negative HSC is c^--*pinched* if

$$(-1 \leq)\ -\frac{\sup_\xi K(\xi)}{\inf_\xi K(\xi)} = c^-,$$

where the infimum and supremum are taken over all non-zero (or unit) tangent vectors across the entire manifold.

In the case of a compact manifold, the infimum and supremum become a minimum and maximum, respectively, due to compactness.

2.6 Ricci curvature and scalar curvature

For a Kähler metric, the *Ricci curvature* $R_{i\bar{j}}$ can be defined as the following trace of the curvature tensor:

$$R_{i\bar{j}} = \sum_{k,l} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}.$$

Positivity or negativity properties of the HSC of a Kähler metric do not necessarily transfer to the Ricci curvature of the same metric. Nevertheless, there is a beautiful integral formula due to Berger (see Lemma 3.4) which expresses the *scalar curvature* τ of a Kähler metric as an integral of the HSC, while the standard definition is as the trace of the Ricci curvature:

$$\tau = \sum_{i,j} g^{i\bar{j}} R_{i\bar{j}} = \sum_{i,j,k,l} g^{i\bar{j}} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}.$$

We would like to cite some of the recent advances here, regarding the following conjecture on the projective manifolds.

Conjecture 2.1. *Let M be a projective manifold with a Kähler metric of negative HSC. Then its canonical line bundle K_M is ample.*

Heier-Lu-Wong proved the above conjecture for a smooth projective threefold in [HLW10, Theorem 1.1]. Wu-Yau proved the conjecture for a higher dimensional projective manifold in [WY16a, Theorem 2]. Later, Tosatti-Yang ([TY16]) proved the conjecture for a compact Kähler manifold. Diverio-Trapani ([DT16]) and Wu-Yau ([WY16b]) subsequently proved it for a compact Kähler manifold of quasi-negative HSC.

Chapter 3

HSC Pinching of Hirzebruch Surfaces

The main result of this chapter is that, for each $n \in \{1, 2, 3, \ldots\}$, there exists a Hodge metric on the n-th Hirzebruch surface whose positive HSC is $\frac{1}{(1+2n)^2}$-pinched. The type of metric under consideration was first studied by Hitchin in this context. In order to address the case $n = 0$, we prove a general result on the pinching of the HSC of the product metric on the product of two Hermitian manifolds M and N of positive HSC.

3.1 Introduction

It is a well-known fact that the Fubini-Study metric on a complex projective space of arbitrary dimension has constant HSC equal to 4. However, in general, few examples are known of compact complex manifolds which carry a Hermitian metric of positive HSC, let alone a Hermitian metric with positively pinched HSC. A notable exception form the irreducible Hermitian symmetric spaces of compact type, whose pinching constants for the HSC are listed in [Che77, Table I] (see also the references in that paper). In particular, the geometry and curvature of fibrations and even fiber bundles are poorly understood in this respect.

In this chapter, we are primarily interested in the Hirzebruch surfaces $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$, $n \in \{0, 1, 2, \ldots\}$. It was proven by Hitchin in [Hit75] that they do carry a natural metric of positive HSC, but his proof does not yield any pinching constants. Even this nonquantitative positivity result may be considered to be somewhat surprising, as the $\mathbb{F}_n$ do not carry metrics of positive Ricci curvature, except when $n = 0$, or $n = 1$. Our main result is the following pinching theorem for the metrics on $\mathbb{F}_n$ considered in [Hit75].

Theorem 3.1. *Let $\mathbb{F}_n$, $n \in \{1, 2, 3, \ldots\}$, be the n-th Hirzebruch surface. Then there exists a Hodge metric on $\mathbb{F}_n$ whose positive HSC is $\frac{1}{(1+2n)^2}$-pinched.*

We also prove that the numerical values of the pinching constants are optimal in the families of metrics studied by Hitchin. This does however leave open the question if there are other types of metrics on Hirzebruch surfaces with better pinching constants. Recall that an upper bound on the possible value of such pinching constants was given in the paper [BG65], where it was proven that a complete Kähler manifold whose positive HSC is c-pinched with $c > \frac{4}{5}$ is homotopic to a complex projective space.

The proofs work by way of explicit computations, using in particular the method of Lagrange multipliers.

Since we could not find a reference for it, we also include the following pinching theorem for products $M \times N$ of Hermitian manifolds endowed with the product metric.

If $M = N = \mathbb{P}^1$, then this theorem addresses the case of the 0-th Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$, which was not handled in Theorem 3.1. In this case, $c_M = c_N = c_{\mathbb{P}^1} = 1$, $k = 4$, and $\frac{c_M c_N}{c_M + c_N} = \frac{1}{2}$.

Theorem 3.2. *Let M and N be Hermitian manifolds whose positive HSCs are c_M- and c_N-pinched respectively and satisfy*

$$kc_M \leq K_M \leq k \quad \textit{and} \quad kc_N \leq K_N \leq k$$

for a constant $k > 0$. Then the HSC, denoted by K, of the product metric on $M \times N$ satisfies

$$k\frac{c_M c_N}{c_M + c_N} \leq K \leq k$$

and is $\frac{c_M c_N}{c_M + c_N}$-pinched.

Recall that the Hopf Conjecture states that the product of two real two-spheres does not admit a Riemannian metric of positive sectional curvature, so even the case of products as in Theorem 3.2 is not trivial with respect to sectional curvatures.

This chapter is organized as follows. In Section 3.2, we will prove Theorem 3.1 and also derive a corollary giving lower and upper bounds for the scalar curvature of the metrics under investigation. In Section 3.3, we will give an interpretation of the results of our computations in terms of the geometry of Hirzebruch surfaces. In Section 3.4, we will prove Theorem 3.2.

3.2 Proof of Theorem 3.1

Following Hitchin's idea from [Hit75], we recall that on the n-th Hirzebruch surface $\mathbb{F}_n$, there are natural Hermitian metrics defined as follows. Note that these metrics are clearly Kähler and, when the value of the parameter s is rational, even Hodge.

If z_1 is an inhomogeneous coordinate on an open subset of the base space $\mathbb{P}^1$, then a point

$$w \in \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$$

can be represented by coordinates (w_1, w_2) in the fiber direction as

$$w = \big(z_1, w_1(dz_1)^{-n/2}, w_2\big),$$

where $(dz_1)^{-1}$ is to be understood as a section of $T\mathbb{P}^1 = \mathcal{O}_{\mathbb{P}^1}(2)$. After the projectivization, each fiber carries the inhomogeneous coordinate $z_2 = w_2/w_1$. For a positive real number s, the metric

$$\omega_s = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Big(\log(1 + z_1\bar{z}_1) + s\log\big((1 + z_1\bar{z}_1)^n + z_2\bar{z}_2\big)\Big)$$

is globally well-defined on $\mathbb{F}_n$. It is this metric for which we compute the HSC pinching. We also find the choice of s with the optimal value of the pinching constant in the family of metrics parametrized by s.

Remark 3.3. *In [Hit75], the curvature tensor is expressed in terms of a local unitary frame field. In this chapter, we prefer to work in terms of the frame field* $\left\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right\}$ *with respect to the coordinates discussed above, as it seems to lend itself better to our method.*

3.2.1 The case $n \geq 2$

As observed in [Hit75], the fact that $SU(2)$ acts transitively on $\mathbb{P}^1$ as isometries of the Fubini-Study metric and that this action lifts to $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$, implies that we can restrict ourselves to computing the curvature along one fiber, say the one given by $z_1 = 0$. The metric tensor associated to ω_s along this fiber is

$$(g_{i\bar{j}}) = \begin{pmatrix} \frac{1+z_2\bar{z}_2+sn}{1+z_2\bar{z}_2} & 0 \\ 0 & \frac{s}{(1+z_2\bar{z}_2)^2} \end{pmatrix}.$$

From this, we see that an orthonormal basis for $T_{(0,z_2)}\mathbb{F}_n$ is given by the two vectors

$$\sqrt{\frac{1+z_2\bar{z}_2}{1+z_2\bar{z}_2+ns}} \cdot \frac{\partial}{\partial z_1} \quad \text{and} \quad \frac{1+z_2\bar{z}_2}{\sqrt{s}} \cdot \frac{\partial}{\partial z_2}.$$

Therefore, an arbitrary unit tangent vector $\xi \in T_{(0,z_2)}\mathbb{F}_n$ can be written as

$$\xi = c_1\sqrt{\frac{1+z_2\bar{z}_2}{1+z_2\bar{z}_2+ns}} \cdot \frac{\partial}{\partial z_1} + c_2\frac{1+z_2\bar{z}_2}{\sqrt{s}} \cdot \frac{\partial}{\partial z_2},$$

where $c_1, c_2 \in \mathbb{C}$ are such that $|c_1|^2 + |c_2|^2 = 1$. Let $\xi_1 := c_1\sqrt{\frac{1+z_2\bar{z}_2}{1+z_2\bar{z}_2+ns}}$ and $\xi_2 := c_2\frac{1+z_2\bar{z}_2}{\sqrt{s}}$. Based on the formula (2.1) in Chapter 2, the components of the curvature tensor are

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &= \frac{2\big(-n^2sz_2\bar{z}_2 + (1+z_2\bar{z}_2)^2 + n(s+sz_2\bar{z}_2)\big)}{(1+z_2\bar{z}_2)^2}, \\ R_{1\bar{1}2\bar{2}} &= \frac{ns(1+ns-z_2^2\bar{z}_2^2)}{(1+z_2\bar{z}_2)^3(1+ns+z_2\bar{z}_2)}, \\ R_{2\bar{2}2\bar{2}} &= \frac{2s}{(1+z_2\bar{z}_2)^4}, \end{aligned}$$

while the other terms (except those obtained from symmetry) are zero. Substituting the components and values of ξ_1 and ξ_2 into the definition (2.2) of HSC in the direction of ξ gives us

$$\begin{aligned} K(\xi) &= 2\sum_{i,j,k,l=1}^{2} R_{i\bar{j}k\bar{l}}\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l \\ &= 2R_{1\bar{1}1\bar{1}}\xi_1\bar{\xi}_1\xi_1\bar{\xi}_1 + 8R_{1\bar{1}2\bar{2}}\xi_1\bar{\xi}_1\xi_2\bar{\xi}_2 + 2R_{2\bar{2}2\bar{2}}\xi_2\bar{\xi}_2\xi_2\bar{\xi}_2 \\ &= \frac{4\big((1+z_2\bar{z}_2)^2 + ns(1+z_2\bar{z}_2 - nz_2\bar{z}_2)\big)}{(1+z_2\bar{z}_2+ns)^2}|c_1|^4 \\ &\quad + \frac{8n(1+ns-z_2^2\bar{z}_2^2)}{(1+z_2\bar{z}_2+ns)^2}|c_1|^2|c_2|^2 + \frac{4}{s}|c_2|^4. \end{aligned}$$

Since the above expression only depends on the modulus squared of z_2, we let $r := z_2\bar{z}_2$. Also, we let $a := |c_1|^2$ and $b := |c_2|^2$, satisfying $a+b = 1$ and $a, b \in [0,1]$. Hence, for fixed values of r and s, the HSC takes the form of a degree two homogeneous polynomial in a and b with real coefficients:

$$K_{r,s}(a,b) = \frac{4\big((1+r)^2 + ns(1+r-nr)\big)}{(1+r+ns)^2}a^2 + \frac{8n(1+ns-r^2)}{(1+r+ns)^2}ab + \frac{4}{s}b^2. \tag{3.1}$$

We write $\alpha := \frac{4\left((1+r)^2+ns(1+r-nr)\right)}{(1+r+ns)^2}$, $\beta := \frac{8n(1+ns-r^2)}{(1+r+ns)^2}$, and $\gamma := \frac{4}{s}$ for the coefficients.

In order to find the pinching constant for the metric ω_s, we need to minimize and maximize

$$K_{r,s}(a,b) = \alpha a^2 + \beta ab + \gamma b^2$$

for fixed s, subject to the constraint $a + b - 1 = 0$. To do so, we first also fix r and set up the Lagrange Multiplier equations:

$$\frac{\partial}{\partial a}K_{r,s}(a,b) = \lambda, \ \frac{\partial}{\partial b}K_{r,s}(a,b) = \lambda, \ a+b-1=0.$$

Solving this system of equations for a, b yields a unique stationary solution

$$\begin{aligned} a_0 &= \frac{2\gamma - \beta}{2(\gamma-\beta+\alpha)} = \frac{(1+r)(1+ns)}{1+s-(-1+n)ns^2+r(1+s+2ns)}, \\ b_0 &= \frac{2\alpha-\beta}{2(\gamma-\beta+\alpha)} = \frac{s(-1+n-r-nr-ns+n^2s)}{-1-r-s-rs-2nrs-ns^2+n^2s^2}. \end{aligned}$$

Substituting these values into equation (3.1) gives us

$$\begin{aligned} &K_{r,s}(a_0,b_0) \\ &= 4 \cdot \frac{3r^2(1+ns)+3r(1+ns)^2-r^3(-1+n^2s)-(1+ns)^2(-1-ns+n^2s)}{(1+r+ns)^2\big(1+s-(-1+n)ns^2+r(1+s+2ns)\big)}. \end{aligned}$$

We shall now find lower and upper bounds for the HSC in the following three cases:

1. *For $a = a_0$ and $b = b_0$*: For a fixed value of s, define $f_s : [0,\infty) \to \mathbb{R}$ as

   $$f_s(r) := K_{r,s}(a_0,b_0).$$

   A computation yields that $f_s'(r) = 0$ if and only if $r = -1 \notin (0,\infty)$ (which we may disregard) or

   $$r = r_0 := \frac{(n-1)(1+ns)}{1+n},$$

   which is in $(0,\infty)$ under the assumption $n \geq 2$. Note

   $$f_s(r_0) = \frac{4-s(n-1)^2}{1+ns}.$$

At the endpoints of the interval $[0,\infty)$, we see that

$$f_s(0) = \frac{4(1+ns-n^2s)}{1+s-(n-1)ns^2}, \quad \text{and} \quad \lim_{r\to\infty} f_s(r) = \frac{4-4n^2s}{1+s+2ns}.$$

The latter expression makes it clear that we need to choose $s < \frac{1}{n^2}$ in order to obtain positive HSC. Furthermore, for $s < \frac{1}{n^2}$,

$$\frac{4(1+ns-n^2s)}{1+s-(n-1)ns^2} - \frac{4-4n^2s}{1+s+2ns} = \frac{4s(3n-s(2n^3-3n^2)-s^2(n^4-n^3))}{(1+s+2ns)\Big(1+s\big(1-s(n^2-n)\big)\Big)} > 0,$$

and

$$\frac{4-s(n-1)^2}{1+ns} - \frac{4(1+ns-n^2s)}{1+s-(n-1)ns^2} = -\frac{s(n-1)^2\big(3+s(n-1)\big)}{\big(-1+s(n-1)\big)(1+ns)} > 0.$$

Thus,

$$\frac{4-s(n-1)^2}{1+ns} > \frac{4(1+ns-n^2s)}{1+s-(n-1)ns^2} > \frac{4-4n^2s}{1+s+2ns}.$$

2. *For $a=0$ and $b=1$:* The HSC value is $K_{r,s}(0,1) = \frac{4}{s}$, which is independent of r.

3. *For $a=1$ and $b=0$:* The HSC value is

$$h_s(r) := K_{r,s}(1,0) = \frac{4((1+r)^2+ns(1+r-nr))}{(1+r+ns)^2}.$$

In the interval $(0,\infty)$, we have that $h_s'(r)=0$ if and only if

$$r = r_0 = \frac{(n-1)(1+ns)}{1+n} \quad \big(\in (0,\infty) \text{ when } n \geq 2\big),$$

with

$$h_s(r_0) = \frac{4 - s(n-1)^2}{1+ns}.$$

Note that this is the same r_0 as above, although we see no clear geometric reason for this coincidence. At the endpoints, we have

$$h_s(0) = \frac{4}{1+ns}, \quad \text{and} \quad \lim_{r\to\infty} h_s(r) = 4.$$

Clearly, we have

$$4 > \frac{4}{1+ns} > \frac{4 - s(n-1)^2}{1+ns}.$$

Combining the three cases above, we have for $n \geq 2$:

$$\frac{4}{s} > 4 > \frac{4}{1+ns} > \frac{4-s(n-1)^2}{1+ns} > \frac{4(1+ns-n^2s)}{1+s-(n-1)ns^2} > \frac{4-4n^2s}{1+s+2ns}.$$

Hence, the smallest and largest values attained by the HSC are

$$\lim_{r\to\infty} f_s(r) = \frac{4-4n^2s}{1+s+2ns} \quad \text{and} \quad \frac{4}{s},$$

respectively.

Finally, in order to find the value of s with the best pinching constant (see 2.5.1 for the definition of pinching constant), we define a function

$$p : \left(0, \frac{1}{n^2}\right) \to (0,1),\ p(s) := \frac{\min_\xi K_s(\xi)}{\max_\xi K_s(\xi)} = \frac{\frac{4-4n^2s}{1+s+2ns}}{\frac{4}{s}} = \frac{s(1-n^2s)}{1+s+2ns},$$

where the minimum and maximum are taken over all non-zero (or unit) tangent vectors across the entire manifold and the index s indicates that the HSC is computed with respect to the metric with the parameter value s. This is the function which we want to maximize. We see that $p'(s) = 0$ if and only if $s = -\frac{1}{n} \notin \left(0, \frac{1}{n^2}\right)$ or $s = \frac{1}{2n^2+n} \in \left(0, \frac{1}{n^2}\right)$. Elementary calculus tells us that p has a global maximum at $\frac{1}{2n^2+n}$. Hence, with $s = \frac{1}{2n^2+n}$, we get the optimal pinching of

$$p\left(\frac{1}{2n^2+n}\right) = \frac{1}{(1+2n)^2}.$$

3.2.2 The case $n = 1$

In the case when $n = 1$, the functions f_s and h_s have their stationary points at the boundary point $r = 0$. However, our reasoning still goes through almost verbatim and yields the expected pinching constant $\frac{1}{9}$ for $s = \frac{1}{3}$.

3.2.3 A remark on scalar curvature

The following formula due to [Ber66, Lemme 7.4] expresses the scalar curvature of a Kähler manifold as an integral of the HSC.

Lemma 3.4. *Let M be an m-dimensional Kähler manifold. Then the scalar curvature τ satisfies at every point $P \in M$:*

$$\tau(P) = \frac{m(m+1)}{4\,\mathrm{vol}(S_P^{2m-1})} \int_{\xi \in S_P^{2m-1}} K(\xi) d\xi,$$

where S_P^{2m-1} denotes the unit sphere inside the tangent space $T_P M$ with respect to the metric, and $d\xi$ is the measure on S_P^{2m-1} induced by the metric.

This lemma yields the following corollary.

Corollary 3.5. *Let τ_s denote the scalar curvature of $\mathbb{F}_n$, $n \in \{1, 2, 3, \ldots\}$, pertaining to the metric ω_s. Then*

$$\frac{3}{2}\min_{\xi} K_s(\xi) = \frac{3}{2} \cdot \frac{4 - 4n^2 s}{1 + s + 2ns} \leq \tau_s \leq \frac{3}{2} \cdot \frac{4}{s} = \frac{3}{2}\max_{\xi} K_s(\xi).$$

In particular, for our optimal choice of $s = \frac{1}{2n^2+n}$, we have

$$\frac{6n(n+1)}{2n^2 + 3n + 1} \leq \tau \leq 12n^2 + 6n.$$

Proof. The proof is immediate from Lemma 3.4 and the bounds for the HSC: Replace the integrand $K_s(\xi)$ by the minimum and maximum, respectively, which we computed, move the constant in front of the integral, cancel $\mathrm{vol}(S_P^{2m-1})$, and let $m = 2$. □

Finally, since the scalar curvature is additive in products equipped with the product metric, and since the scalar curvature of $\mathbb{P}^1$ with the Fubini-Study metric is constant and equal to 2, it is immediately clear that the scalar curvature of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is constant and equal to 4.

3.3 Geometric interpretation of our computations

The Hirzebruch surfaces have a beautiful geometric structure, which is very nicely explained in [GH94, pp. 517–520]. In particular, on the n-th Hirzebruch surface, there is a unique non-singular rational curve E "at infinity" which has self-intersection number $-n$. In terms of our coordinates (z_1, z_2), the curve E is given by $z_2 = \infty$. The fact that

$$\min_{\xi} K_s(\xi) = \lim_{r\to\infty} f_s(r)$$

means that the smallest value of the HSC for each ω_s is attained at a tangent vector attached to a point of E. Note that because of the transitivity of the $SU(2)$ action, this is then true for all points of E. Since the largest value $\frac{4}{s}$ is attained inside every tangent space of $\mathbb{F}_n$, every point $P \in E$ has the property that the tangent space to $\mathbb{F}_n$ at P contains a vector giving the lowest possible HSC and a vector giving the highest possible HSC. In other words, for Hirzebruch surfaces, the notion of the "pinching constant" and the "pointwise pinching constant" are one and the same.

We can still say more about the vectors yielding the extreme values. If we consider a_0 and b_0 as functions of r and set $s = \frac{1}{2n^2+n}$, then

$$\lim_{r\to\infty} a_0 = \frac{1+ns}{1+s+2ns} = \frac{2n}{2n+1}, \text{ and } \lim_{r\to\infty} b_0 = \frac{s(1+n)}{1+s+2ns} = \frac{1+n}{1+3n+2n^2}.$$

For large values of n, the first value is a little less than 1, and the second value is a little larger than 0. This means that the direction of the tangent vector giving the smallest value of the HSC is close, but not equal, to the direction of the tangent space of E, which we think of as the "horizontal" direction. Moreover, the direction of the tangent vector giving the largest value of the HSC is exactly "vertical" and thus almost, but not exactly, perpendicular to the direction giving the smallest value.

3.4 Proof of Theorem 3.2

The proof of Theorem 3.2 consists of computing the HSC of the product metric on the product $M^m \times N^n$, $m, n \in \{1, 2, 3, \ldots\}$, of two Hermitian manifolds with local coordinates $(z_1, \ldots, z_m)$ and $(z_{m+1}, \ldots, z_{m+n})$ around points $p \in M$ and $q \in N$, respectively. Let $g = \sum_{i,j=1}^{m} g_{i\bar{j}}\, dz_i \otimes d\bar{z}_j$, and $h = \sum_{i,j=m+1}^{m+n} h_{i\bar{j}}\, dz_i \otimes d\bar{z}_j$ be Hermitian metrics on M and N, respectively, with positive HSC. Then

$$\sum_{i,j=1}^{m} g_{i\bar{j}}\ dz_i \otimes d\bar{z}_j + \sum_{i,j=m+1}^{m+n} h_{i\bar{j}}\ dz_i \otimes d\bar{z}_j$$

gives the product metric in a neighborhood of $(p, q) \in M \times N$. Since the $g_{i\bar{j}}$'s are functions of only $z_1, \ldots, z_m$ and the $h_{i\bar{j}}$'s are functions of only $z_{m+1}, \ldots, z_{m+n}$, we obtain

$$R_{i\bar{j}k\bar{l}} = \begin{cases} -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{a,b=1}^{m} g^{a\bar{b}} \frac{\partial g_{i\bar{a}}}{\partial z_k} \frac{\partial g_{b\bar{j}}}{\partial \bar{z}_l}, & 1 \le i, j, k, l \le m \\ \\ -\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{a,b=m+1}^{m+n} h^{a\bar{b}} \frac{\partial h_{i\bar{a}}}{\partial z_k} \frac{\partial h_{b\bar{j}}}{\partial \bar{z}_l}, & m+1 \le i, j, k, l \le m+n \\ \\ 0, & \text{otherwise.} \end{cases}$$

Let $\xi = \sum_{i=1}^{m+n} \xi_i\ \frac{\partial}{\partial z_i}$ be a unit tangent vector in $T_{(p,q)}(M \times N)$. Then the HSC on $M \times N$ along ξ is

$$\begin{aligned} K(\xi) = 2 &\sum_{i,j,k,l=1}^{m} \left(-\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{a,b=1}^{m} g^{a\bar{b}} \frac{\partial g_{i\bar{a}}}{\partial z_k} \frac{\partial g_{b\bar{j}}}{\partial \bar{z}_l} \right) \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l \\ &+ 2 \sum_{i,j,k,l=m+1}^{m+n} \left(-\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{a,b=m+1}^{m+n} h^{a\bar{b}} \frac{\partial h_{i\bar{a}}}{\partial z_k} \frac{\partial h_{b\bar{j}}}{\partial \bar{z}_l} \right) \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l. \end{aligned}$$

The two sums on the right hand side above are the numerators of the HSCs on M and N with respect to the tangent vectors $(\xi_1, \ldots, \xi_m) \in T_pM$ and $(\xi_{m+1}, \ldots, \xi_{m+n}) \in$

T_qN, respectively, both of which are positive. Thus,

$$K(\xi) > 0.$$

In order to find the pinching constant, we need to take into consideration the (non-zero) norms of $(\xi_1, \dots, \xi_m) \in T_pM$ and $(\xi_{m+1}, \dots, \xi_{m+n}) \in T_qN$ with respect to the respective metrics in the two spaces, as follows:

$$\begin{aligned}
K(\xi) &= \sum_{i,k,j,l=1}^{m} 2R_{i\bar{j}k\bar{l}}\ \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l + \sum_{i,k,j,l=m+1}^{m+n} 2R_{i\bar{j}k\bar{l}}\ \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l \\
&= \frac{\sum_{i,k,j,l=1}^{m} 2R_{i\bar{j}k\bar{l}}\ \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l}{\sum_{i,j,k,l=1}^{m} g_{i\bar{j}}g_{k\bar{l}}\ \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l} \cdot \sum_{i,j,k,l=1}^{m} g_{i\bar{j}}g_{k\bar{l}}\ \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l \\
&\quad + \frac{\sum_{i,k,j,l=m+1}^{m+n} 2R_{i\bar{j}k\bar{l}}\ \xi\bar{\xi}_j\xi_k\bar{\xi}_l}{\sum_{i,j,k,l=m+1}^{m+n} h_{i\bar{j}}h_{k\bar{l}}\ \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l} \cdot \sum_{i,j,k,l=m+1}^{m+n} h_{i\bar{j}}h_{k\bar{l}}\ \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l \\
&= K_M \cdot y^2 + K_N \cdot (1-y)^2,
\end{aligned}$$

where K_M is the HSC of M along $(\xi_1, \dots, \xi_m)$, K_N the HSC of N along $(\xi_{m+1}, \dots, \xi_{m+n})$ and $y = \sum_{i,j}^{m} g_{i\bar{j}}\ \xi_i\bar{\xi}_j$.

Since ξ is a unit tangent vector in $T_{(p,q)}(M \times N)$, i.e., $\sum_{i,j=1}^{m} g_{i\bar{j}}\ \xi_i\bar{\xi}_j + \sum_{i,j=m+1}^{m+n} h_{i\bar{j}}\ \xi_i\bar{\xi}_j = 1$, we have

$$\sum_{i,j=m+1}^{m+n} h_{i\bar{j}}\ \xi_i\bar{\xi}_j = 1 - \sum_{i,j=1}^{m} g_{i\bar{j}}\ \xi_i\bar{\xi}_j = 1 - y.$$

Furthermore, the assumption

$$kc_M \le K_M \le k \quad \text{and} \quad kc_N \le K_N \le k$$

provides the following inequality:

$$F(y) := kc_M y^2 + kc_N(1-y)^2 \leq \; K_M y^2 + K_N(1-y)^2 \; \leq ky^2 + k(1-y)^2 =: \widetilde{F}(y).$$

Finally, elementary calculus yields

$$\min_{0\leq y\leq 1} F(y) = k\frac{c_M c_N}{c_M + c_N}$$

and

$$\max_{0\leq y\leq 1} \widetilde{F}(y) = k.$$

In particular,

$$k\frac{c_M c_N}{c_M + c_N} \leq K(\xi) \leq k,$$

and the pinching constant for the HSC on the product space is obtained as

$$c_{M\times N} = \frac{\inf_\xi K(\xi)}{\sup_\xi K(\xi)} = \frac{c_M c_N}{c_M + c_N}.$$

Chapter 4

Pinching for Isotrivial Families of Curves Over a Curve in Negative Curvature

A family of smooth curves $f: \mathcal{F} \to C$ over a smooth curve C is called *isotrivial* if for any two points $a,\ b \in C$, the fibers $\mathcal{F}_a$ and $\mathcal{F}_b$ are isomorphic to each other, i.e., $\mathcal{F}_a \cong \mathcal{F}_b \cong F$ for a smooth curve F.

The main result of this chapter is on the negative HSC of isotrivial families of curves over a curve, where the base and the fiber have genus greater than or equal to 2. This result is analogous to Theorem 3.1 on the Hirzebruch surfaces, but it is even better in terms of the pinching constant, as it is independent of the geometry of the family. The precise statement of the theorem is as follows:

Theorem 4.1. *Let $f: \mathcal{F} \to C$ be an isotrivial family of smooth curves over a smooth curve C, such that all the fibers are isomorphic to a smooth curve F. Let the genus* *of both F and C be greater than or equal to 2. Then there exists a Hermitian metric on $\mathcal{F}$ whose HSC is negative and is $-\frac{1}{2}$-pinched.*

There are two crucial points in the proof of Theorem 4.1. One of them requires the existence of a Hermitian metric of negative HSC on the product of two smooth curves of genus greater than or equal to 2. For the second point, we need the existence of a metric of negative HSC on a space if its covering space carries a metric of negative HSC. Both of these requirements are fulfilled, not only in the 1-dimensional case, but also for the higher dimensional manifolds as shown in Theorem 4.2 and Theorem 4.3, respectively.

Theorem 4.2. *Let M and N be Hermitian manifolds whose negative HSCs are c_M and c_N-pinched, respectively, and satisfy*

$$-k \leq K_M \leq kc_M < 0 \qquad \textit{and} \qquad -k \leq K_N \leq kc_N < 0$$

for a constant $k > 0$. Then the HSC, denoted by K, of the product metric on $M \times N$ satisfies

$$-k \leq K \leq k\frac{c_M c_N}{c_M + c_N} < 0,$$

and is $\frac{c_M c_N}{c_M + c_N}$-pinched. Moreover, the HSC of the product metric on $M \times N$ is semi-negative if the HSC of M or N (or both) is semi-negative.

Clearly, Theorem 4.2 is the negative curvature analog of Theorem 3.2. Moreover, we can prove Theorem 4.2 in the same manner as we proved Theorem 3.2, i.e., by direct computation.

A curve of genus greater than or equal to 2 has constant negative HSC, i.e., the pinching constant is equal to -1. Therefore, the pinching constant $-\frac{1}{2}$ in Theorem

4.1 agrees with the formula of the pinching constant in Theorem 4.2.

Theorem 4.3. *Let M and $\widetilde{M}$ be Hermitian manifolds and $f : \widetilde{M} \to M$ be a finite covering map of degree d. Then $\widetilde{M}$ carries a Hermitian metric of (semi-)negative HSC if and only if M carries a Hermitian metric of (semi-)negative HSC.*

Before proceeding towards the proofs of Theorem 4.3 and Theorem 4.1, we would like to recall a few concepts as mentioned in the following two sections.

4.1 Covering space

Let M and $\widetilde{M}$ be complex manifolds with a holomorphic map $\pi : \widetilde{M} \to M$. Suppose that for every point $p \in M$, there exists an open neighborhood U of p in M, such that the inverse image of U via π is a disjoint union of d open sets in $\widetilde{M}$, i.e.,

$$\pi^{-1}(U) = \bigsqcup_{i=1}^{d} V_i,$$

where V_i are open sets in $\widetilde{M}$ such that U is biholomorphic to V_i for each $i = 1, \ldots, d$. Then $\widetilde{M}$ is called a *finite covering space* of M, and π is called the corresponding *covering map* of *degree* d. $\widetilde{M}$ is called an *infinite covering space* of M if

$$\pi^{-1}(U) = \bigsqcup_{i=1}^{\infty} V_i.$$

In this case, the degree of π is infinite.

If $\widetilde{M}$ is simply connected, then it is called the *universal cover* of M. The adjective "universal" is used to emphasize the fact that if $\pi' : M' \to M$ is another cover of

M, then there exists a (essentially unique) holomorphic map $f : \widetilde{M} \to M'$ such that $\pi' \circ f = \pi$, i.e., the following diagram commutes:

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "f"] \arrow[d, "\pi"'] & M' \arrow[dl, "\pi'"] \\
M &
\end{tikzcd}$$

4.2 Level structures and finite base change for an isotrivial family of curves

Let C be a curve of genus g. A *level n structure* on C is defined to be a symplectic basis $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ for the homology group $H_1(C, \mathbb{Z}/n)$, where *symplectic* means that the intersection pairing on $H_1(C, \mathbb{Z}/n)$ has the following matrix form:

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

In other words, there is a symplectic isomorphism: $(\mathbb{Z}/n)^{2g} \to H^1(C, \mathbb{Z}/n)$.

The moduli space of curves of genus g, with level n structure, is denoted by $\mathcal{M}_g(n)$. If $g \geq 2$ and $n \geq 3$, then $\mathcal{M}_g(n)$ is a *fine* moduli space, i.e., it is a universal parameter space for families of curves of genus g with a level n structure.

Let $f : \mathcal{F} \to C$ be an isotrivial family of smooth curves over a smooth curve C, such that the base C and the fiber F have genus greater than or equal to 2. Also let $\pi : \widetilde{C} \to C$ be a finite covering of C, such that the local system $H^1(\mathcal{F}_c, \mathbb{Z}/n)_{c\in C}$ becomes trivial on $\widetilde{C}$. Then, using the facts that $\mathcal{F}$ is isotrivial, and $\mathcal{M}_g(n)$ is fine, the classifying map $\widetilde{C} \to \mathcal{M}_g(n)$ is constant. Therefore, the pull back of f to $\widetilde{C}$ is trivial, i.e., there exists a biholomorphic map $\phi : F \times \widetilde{C} \to \mathcal{F} \times_C \widetilde{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
F \times \widetilde{C} \arrow[r, "\phi"] \arrow[dr, "\pi_2"'] & \mathcal{F} \times_C \widetilde{C} \arrow[r, "\pi_1"] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\
& \widetilde{C} \arrow[r, "\pi"] & C
\end{tikzcd}$$

where π_1 and π_2 are the projection maps onto the first and second components, respectively. In other words, an isotrivial family of smooth curves over a smooth curve splits by a finite base change, if the base and the fiber have genus greater than or equal to 2.

One may refer to [HM98, Chapters 1 and 2] for a detailed explanation of the moduli spaces of curves with level structures.

4.3 Proof of Theorem 4.3

Let $U \subset M$ be a small open neighborhood of a point $p \in M$. Then, there exist open sets $V_1, V_2, \ldots, V_d$ in $\widetilde{M}$ such that $f^{-1}(U) = V_1 \sqcup V_2 \sqcup \ldots \sqcup V_d$. Suppose $q_i \in V_i$, $i = 1, 2, \ldots, d$, such that $f^{-1}(p) = \{q_1, q_2, \ldots, q_d\}$. The mappings $f_i := f|_{V_i}$ are biholomorphisms between V_i and U for all $i = 1, 2, \ldots, d$. Therefore, for a Hermitian metric G in M, the pullback metric $(f_i)^*(G)$ defined by

$$(f_i)^*(G)(q_i)(X,Y) = G(p)\Big((f_i)_*X, (f_i)_*Y\Big), \qquad X, Y \in T_{q_i}V_i,$$

is a Hermitian metric on V_i, i.e., f_i's are isometries with respect to the metrics G and $(f_i)^*(G)$. Thus, the HSC at p with respect to G is same as the HSC at q_i with respect to $(f_i)^*(G)$.

For the other direction, we notice that $f_i^{-1} : U \to V_i$ are also biholomorphisms for all $i = 1, 2, \ldots, d$. If H is a Hermitian metric on $\widetilde{M}$, then using the same argument as above, we obtain a Hermitian metric $(f_i^{-1})^*(H|_{V_i})$ on U, such that f_i^{-1}'s are also isometries with respect to the metrics $H|_{V_i}$ and $(f_i^{-1})^*(H|_{V_i})$. Therefore, the HSC at q_i with respect to $H|_{V_i}$ is same as the HSC at p with respect to $(f_i^{-1})^*(H|_{V_i})$. We define a metric $\widehat{H}$ on U by taking the average of all these pullback metrics:

$$\widehat{H} = \frac{1}{d}\sum_{i=1}^{d}(f_i^{-1})^*(H|_{V_i}).$$

If the HSC at q_i with respect to $H|_{V_i}$ is (semi-) negative, then the HSC at p with respect to $(f_i^{-1})^*(H|_{V_i})$ is also (semi-) negative. Then, the HSC at p with respect to $\widehat{H}$ is (semi-) negative because of a repetitive application of [Wu73, Theorem 1]. We may refer to $\widehat{H}$ as the *pushforward metric* of H via the covering map f.

4.4 Proof of Theorem 4.1

For the isotrivial family of curves $f : \mathcal{F} \to C$, it is given that the base C and the fiber F have genus greater than or equal to 2. Section 4.2 implies the existence of a finite cover $\pi : \widetilde{C} \to C$ of C and a biholomorphic map $\phi : F \times \widetilde{C} \to \mathcal{F} \times_C \widetilde{C}$ such that the following diagram commutes

$$\begin{tikzcd}
F \times \widetilde{C} \arrow[r, "\phi"] \arrow[dr, "\pi_2"'] & \mathcal{F} \times_C \widetilde{C} \arrow[r, "\pi_1"] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\
& \widetilde{C} \arrow[r, "\pi"] & C
\end{tikzcd}$$

where π_1 and π_2 are the projection maps onto the first and second components, respectively.

Let d be the degree of the covering map π. If $g(C)$ and $g(\widetilde{C})$ denote the genus of C and $\widetilde{C}$, respectively, then according to the Riemann-Hurwitz formula,

$$g(\widetilde{C}) = \frac{b}{2} + d\big(g(C) - 1\big) + 1,$$

where b is the total branching order of π (which in this case is zero, since we are considering an unramified covering space). It is clear from the Riemann-Hurwitz formula that the genus of C being greater than or equal to 2 implies that the genus of the covering space $\widetilde{C}$ is also greater than or equal to 2. Moreover, the fact that both F and $\widetilde{C}$ are curves of genus greater than or equal to 2 implies that there exist Hermitian metrics (Poincaré metric) on F and $\widetilde{C}$ of constant negative HSC. Therefore using Theorem 4.2, $F \times \widetilde{C}$ has a Hermitian metric (product metric) of negative HSC. Finally, Theorem 4.3 provides a metric of negative HSC on $\mathcal{F}$ by pushing forward the product metric on $F \times \widetilde{C}$ to $\mathcal{F}$ via $\pi_1 \circ \phi$.

4.4.1 Pinching

Let $p \in \mathcal{F}$ and U be a neighborhood of p such that $(\pi_1 \circ \phi)^{-1}(p) = \{q_1, \ldots, q_d\}$ and $(\pi_1 \circ \phi)^{-1}(U) = \bigsqcup_{i=1}^{d} V_i$, where $q_i \in V_i$ and V_i's are pairwise disjoint open sets in $F \times \widetilde{C}$. The map $\phi_i := (\pi_1 \circ \phi)|_{V_i}$ is a biholomorphism for each $i = 1, \ldots, d$. Let H denotes the product metric on $F \times \widetilde{C}$ which is obtained from the Poincaré metric on F and $\widetilde{C}$. Then $\left(\phi_i^{-1}\right)^*(H|_{V_i})$ is a Hermitian metric on U for each $i = 1, \ldots, d$. Since the isometry group of the Poincare disk acts transitively on it, any two points on a hyperbolic Riemann surface have isometric neighborhoods. This implies that the Hermitian metrics $\left(\phi_i^{-1}\right)^*(H|_{V_i})$ and $\left(\phi_j^{-1}\right)^*(H|_{V_j})$, defined on U, are isometric for all $i, j = 1, \ldots, d$. Therefore, the pushforward metric of H is given by

$$(\pi_1 \circ \phi)_* H = \frac{1}{d} \sum_{i=1}^{d} \left(\phi_i^{-1}\right)^*(H|_{V_i}) = \left(\phi_i^{-1}\right)^*(H|_{V_i})$$

for all $i = 1, \ldots, d$. This implies that $\pi_1 \circ \phi$ is an isometry, and the HSC of $\mathcal{F}$ with respect to $(\pi_1 \circ \phi)_* H$ is same as the HSC of $F \times \widetilde{C}$ with respect to H.

The HSC of F and $\widetilde{C}$ with respect to the Poincaré metric is -1 which is same as the pinching constants for the two curves. Using the formula for the pinching constant in Theorem 4.2, the HSC of $F \times \widetilde{C}$ and hence of $\mathcal{F}$ is $-\frac{1}{2}$-pinched with respect to H and $(\pi_1 \circ \phi)_* H$, respectively.

Chapter 5

Hermitian Metrics of Positive HSC on Fibrations

We saw in Chapter 3 and Chapter 4 that the signs of the HSCs on the base and fibers of the Hirzebruch surfaces and isotrivial families of curves carry over to the HSC on the entire manifold. Moreover, Cheung proved in [Che89, Theorem 1] that negative HSCs on the base and fibers of a compact fibration also carry over to the HSC on the entire fibration. We shall see in this chapter that the analogous result for a compact fibration in the case of positive HSC also holds true. The following is the main result of this chapter, which could perhaps also be considered to be the main theorem of this dissertation.

Theorem 5.1. *Let $\pi : X \to Y$ be a compact fibration. Assume that Y has a Hermitian metric of positive HSC, and there exists a smooth family of Hermitian metrics on the fibers which all have positive HSC. Then there exists a Hermitian metric on X with positive HSC everywhere.*

Note that the case of projectivized vector bundles was treated in [AHZ16].

We need the following two lemmas for the proof of Theorem 5.1. Overall, we follow the results and proofs in [Che89], although due to the different natures of positive and negative HSC, certain key aspects had to be treated differently.

Lemma 5.2. *Let M be an n-dimensional Hermitian manifold, and G be a Hermitian metric on M. Let $R_{i\bar{j}k\bar{l}}$ be the components of the curvature tensor with respect to G for $i, j, k, l = 1, \ldots, n$. Suppose the following is true at a point $p \in M$ for some positive constants K_0, K_1, K_2, and a natural number $s < n$:*

1.
$$\sum_{i,j,k,l=1}^{s} R_{i\bar{j}k\bar{l}}(p)\, \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l \geq K_0 \sum_{i,j=1}^{s} \xi_i\bar{\xi}_i\xi_j\bar{\xi}_j$$
for all $\xi_i \in \mathbb{C}$, $\quad i = 1, 2, \ldots, s$.

2.
$$|R_{i\bar{j}k\bar{l}}(p)| < K_1$$
whenever $\min(i, j, k, l) \leq s$.

3.
$$\sum_{\alpha,\beta,\gamma,\delta=s+1}^{n} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(p)\, \xi_\alpha\bar{\xi}_\beta\xi_\gamma\bar{\xi}_\delta \geq K_2 \sum_{\alpha,\beta=s+1}^{n} \xi_\alpha\bar{\xi}_\alpha\xi_\beta\bar{\xi}_\beta$$
for any $\xi_\alpha \in \mathbb{C}$, $\quad \alpha = s+1, s+2, \ldots, n$.

Then there exists a positive constant $\mathcal{K}$ depending only on K_0/K_1 such that if $K_2/K_1 \geq \mathcal{K}$, then G has positive HSC at the point p.

Lemma 5.3. *Let M be an n-dimensional complex manifold with two Hermitian metrics G and H defined on it. Suppose that the metric H has positive HSC at a point $p \in M$. Then $G + \lambda H$ also has positive HSC at p for λ large enough.*

5.1 Proof of Lemma 5.2

Suppose the Hermitian metric G on M is given by

$$G = \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz_i \otimes d\bar{z}_j.$$

Since we are only interested in the sign of the HSC, it suffices to check the numerator of (2.2) for positive sign at p in the direction of $\xi = (\xi_1, \ldots, \xi_n)$ with respect to the Hermitian metric G. Applying the hypothesis of the lemma, we obtain

$$\begin{aligned}\sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}}(p)\, \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l \geq K_0 \sum_{i,j=1}^{s} \xi_i\bar{\xi}_i\xi_j\bar{\xi}_j - 4K_1 \sum_{\alpha,\beta,\gamma=s+1}^{n} \sum_{i=1}^{s} |\xi_i||\xi_\alpha||\xi_\beta||\xi_\gamma| \\ - 6K_1 \sum_{\alpha,\beta=s+1}^{n} \sum_{i,j=1}^{s} |\xi_i||\xi_j||\xi_\alpha||\xi_\beta| \\ - 4K_1 \sum_{\alpha=s+1}^{n} \sum_{i,j,k=1}^{s} |\xi_i||\xi_j||\xi_k||\xi_\alpha| + K_2 \sum_{\alpha,\beta=s+1}^{n} \xi_\alpha\bar{\xi}_\alpha\xi_\beta\bar{\xi}_\beta\end{aligned} \tag{5.1}$$

for any choice of $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$. The coefficients 4, 6, 4 in the above expression are obtained from summing of the indices. For any choice of positive numbers a, b, c, d, we have:

$$\begin{aligned}|\xi_i||\xi_\alpha||\xi_\beta||\xi_\gamma| &\leq a^2|\xi_i|^4 + \frac{1}{a^2}|\xi_\alpha|^4 + |\xi_\beta|^2|\xi_\gamma|^2, \\ |\xi_i||\xi_j||\xi_\alpha||\xi_\beta| &\leq b^2|\xi_i|^2|\xi_j|^2 + \frac{1}{b^2}|\xi_\alpha|^2|\xi_\beta|^2, \\ |\xi_i||\xi_j||\xi_k||\xi_\alpha| &\leq c^2|\xi_i|^2|\xi_j|^2 + \frac{d^2}{c^2}|\xi_k|^4 + \frac{1}{c^2d^2}|\xi_\alpha|^4.\end{aligned} \tag{5.2}$$

Substituting the inequalities from (5.2) in (5.1), we obtain

$$\begin{aligned}
&\sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}}(p)\,\xi_i\overline{\xi}_j\xi_k\overline{\xi}_l \\
\geq\ & K_0\sum_{i,j=1}^{s}|\xi_i|^2|\xi_j|^2 - 4K_1\sum_{\alpha,\beta,\gamma=s+1}^{n}\sum_{i=1}^{s}\left(a^2|\xi_i|^4+\frac{1}{a^2}|\xi_\alpha|^4+|\xi_\beta|^2|\xi_\gamma|^2\right) \\
&- 6K_1\sum_{\alpha,\beta=s+1}^{n}\sum_{i,j=1}^{s}\left(b^2|\xi_i|^2|\xi_j|^2+\frac{1}{b^2}|\xi_\alpha|^2|\xi_\beta|^2\right) - 4K_1\sum_{\alpha=s+1}^{n}\sum_{i,j,k=1}^{s}\left(c^2|\xi_i|^2|\xi_j|^2\right. \\
&\left.+\frac{d^2}{c^2}|\xi_k|^4+\frac{1}{c^2d^2}|\xi_\alpha|^4\right) + K_2\sum_{\alpha,\beta=s+1}^{n}|\xi_\alpha|^2|\xi_\beta|^2 \\
=\ & K_0\sum_{i,j=1}^{s}|\xi_i|^2|\xi_j|^2 - K_1\left(4a^2(n-s)^3\sum_{i=1}^{s}|\xi_i|^4+\frac{4}{a^2}s(n-s)^2\sum_{\alpha=s+1}^{n}|\xi_\alpha|^4\right. \\
&+4s(n-s)\sum_{\alpha,\beta=s+1}^{n}|\xi_\alpha|^2|\xi_\beta|^2+6b^2(n-s)^2\sum_{i,j=1}^{s}|\xi_i|^2|\xi_j|^2+\frac{6}{b^2}s^2\sum_{\alpha,\beta=s+1}^{n}|\xi_\alpha|^2|\xi_\beta|^2 \\
&\left.+4c^2s(n-s)\sum_{i,j=1}^{s}|\xi_i|^2|\xi_j|^2+\frac{4d^2}{c^2}s^2(n-s)\sum_{i=1}^{s}|\xi_i|^4+\frac{4}{c^2d^2}s^3\sum_{\alpha=s+1}^{n}|\xi_\alpha|^4\right) \\
&+K_2\sum_{\alpha,\beta=s+1}^{n}|\xi_\alpha|^2|\xi_\beta|^2.
\end{aligned}$$

Using $\sum_{i=1}^{s}|\xi_i|^4 \leq \sum_{i,j=1}^{s}|\xi_i|^2|\xi_j|^2$ and $\sum_{\alpha=s+1}^{n}|\xi_\alpha|^4 \leq \sum_{\alpha,\beta=s+1}^{n}|\xi_\alpha|^2|\xi_\beta|^2$, we obtain

$$\begin{aligned}
\sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}}(p)\,\xi_i\overline{\xi}_j\xi_k\overline{\xi}_l \geq\ & K_0\sum_{i,j=1}^{s}|\xi_i|^2|\xi_j|^2 - K_1\left(\left(4a^2(n-s)^3+6b^2(n-s)^2\right.\right. \\
&\left.+4c^2s(n-s)+\frac{4d^2}{c^2}s^2(n-s)\right)\sum_{i,j=1}^{s}|\xi_i|^2|\xi_j|^2 \\
&\left.+\left(\frac{4}{a^2}s(n-s)^2+4s(n-s)+\frac{6}{b^2}s^2+\frac{4}{c^2d^2}s^3\right)\sum_{\alpha,\beta=s+1}^{n}|\xi_\alpha|^2|\xi_\beta|^2\right) \\
&+K_2\sum_{\alpha,\beta=s+1}^{n}|\xi_\alpha|^2|\xi_\beta|^2.
\end{aligned}$$

We may choose a, b, c, d such that

$$4a^2(n-s)^3 + 6b^2(n-s)^2 + 4c^2 s(n-s) + \frac{4d^2}{c^2} s^2(n-s) \leq \frac{1}{2}\frac{K_0}{K_1}.$$

Let $\mathcal{K} = \frac{4}{a^2} s(n-s)^2 + 4s(n-s) + \frac{6}{b^2} s^2 + \frac{4}{c^2 d^2} s^3$. Note that since the choice of a, b, c, d is based on K_0/K_1, therefore $\mathcal{K}$ too depends only on K_0/K_1. Then for such a choice of a, b, c and d,

$$\begin{aligned}\sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}}(p)\, \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l \geq{}& K_0 \sum_{i,j=1}^{s} |\xi_i|^2 |\xi_j|^2 - K_1 \left(\frac{K_0}{2K_1} \sum_{i,j=1}^{s} |\xi_i|^2 |\xi_j|^2 + \mathcal{K} \sum_{\alpha,\beta=s+1}^{n} |\xi_\alpha|^2 |\xi_\beta|^2 \right) \\ &+ K_2 \sum_{\alpha,\beta=s+1}^{n} |\xi_\alpha|^2 |\xi_\beta|^2 \\ ={}& \frac{K_0}{2} \sum_{i,j=1}^{s} |\xi_i|^2 |\xi_j|^2 + \big(K_2 - K_1 \mathcal{K}\big) \sum_{\alpha,\beta=s+1}^{n} |\xi_\alpha|^2 |\xi_\beta|^2.\end{aligned}$$

If $K_2 - K_1\mathcal{K} \geq 0$, i.e., $\mathcal{K} \leq K_2/K_1$, then clearly $2\sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}}(p)\, \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l$ is positive, which is the numerator of the HSC in the direction of a tangent vector $(\xi_1, \ldots, \xi_n)$ as given in (2.2). Since the denominator of (2.2) is always positive, we conclude that the HSC at p with respect to G is positive in the direction of $(\xi_1, \ldots, \xi_n)$, if the above condition is satisfied $\big($i.e., $\mathcal{K} \leq K_2/K_1\big)$.

5.2 Proof of Lemma 5.3

For the given point $p \in M$ and a unit tangent vector t at p, we choose local coordinates $(z_1, \ldots, z_n)$ at p which satisfy the conditions in [Wu73, Lemma 3] with respect to H, i.e.,

1. $z_1(p) = \ldots = z_n(p) = 0$.

2. If $H = \sum_{i,j=1}^n h_{i\bar{j}}\ dz_i \otimes d\overline{z}_j$, then $h_{i\bar{j}}(p) = \delta_{i\bar{j}}$ and

$$\frac{\partial h_{i\bar{j}}}{\partial z_n}(p) = \frac{\partial h_{i\bar{j}}}{\partial \overline{z}_n}(p) = 0,$$

   for $1 \leq i,\ j \leq n$.

3. $t = \frac{\partial}{\partial z_n}(p)$.

Let $M' = \{z_1 = \ldots = z_{n-1} = 0\}$ be a 1-dimensional complex submanifold of M tangent to t. The Gaussian curvature of M' at p, with respect to the induced metric $H'\big(= H|_{M'} = h_{n\bar{n}}\ dz_n \otimes d\overline{z}_n\big)$, equals the HSC at p with respect to H in the direction of t, denoted by $K(H,t)(p)$ ([Wu73, Lemma 4]). Moreover, if $G = \sum_{i,j=1}^n g_{i\bar{j}}\, dz_i \otimes d\overline{z}_j$, then the induced metric of G on M' is given by $G' = G|_{M'} = g_{n\bar{n}} dz_n \otimes d\overline{z}_n$. Let us denote $g = g_{n\bar{n}}$, $h = h_{n\bar{n}}$, and $z = z_n$. Then, $G' + \lambda H' = (g + \lambda h)\, dz \otimes d\overline{z}$ is the induced metric of $G + \lambda H$ on M'. The HSC at p with respect to $G' + \lambda H'$ is given by

$$\begin{aligned}
K\big(G' + \lambda H'\big)(p) &= \frac{2}{\big(g(p) + \lambda h(p)\big)^3}\Bigg(- \big(g(p) + \lambda h(p)\big)\bigg(\frac{\partial^2 g}{\partial z \partial \overline{z}}(p) + \lambda \frac{\partial^2 h}{\partial z \partial \overline{z}}(p)\bigg) \\
&\quad + \bigg(\frac{\partial g}{\partial z}(p) + \lambda \frac{\partial h}{\partial z}(p)\bigg)\bigg(\frac{\partial g}{\partial \overline{z}}(p) + \lambda \frac{\partial h}{\partial \overline{z}}(p)\bigg)\Bigg) \\
&= \frac{2}{\big(g(p) + \lambda h(p)\big)^3}\Bigg(- g(p)\frac{\partial^2 g}{\partial z \partial \overline{z}}(p) + \frac{\partial g}{\partial z}(p)\frac{\partial g}{\partial \overline{z}}(p) \\
&\quad - \lambda^2 h(p)\frac{\partial^2 h}{\partial z \partial \overline{z}}(p) + \lambda^2 \frac{\partial h}{\partial z}(p)\frac{\partial h}{\partial \overline{z}}(p) - \lambda h(p)\frac{\partial^2 g}{\partial z \partial \overline{z}}(p) \\
&\quad - \lambda g(p)\frac{\partial^2 h}{\partial z \partial \overline{z}}(p) + \lambda \frac{\partial g}{\partial z}(p)\frac{\partial h}{\partial \overline{z}}(p) + \lambda \frac{\partial h}{\partial z}(p)\frac{\partial g}{\partial \overline{z}}(p)\Bigg),
\end{aligned}$$

which simplifies to

$$
\begin{aligned}
&K\big(G' + \lambda H'\big)(p) \\
&= \frac{1}{\big(g(p)+\lambda h(p)\big)^3}\Bigg(g(p)^3 K(G')(p) + \lambda^2 h(p)^3 K(H')(p) + 2\lambda\Bigg(-h(p)\frac{\partial^2 g}{\partial z \partial \bar{z}}(p) \\
&\quad - g(p)\frac{\partial^2 h}{\partial z \partial \bar{z}}(p) + \frac{\partial g}{\partial z}(p)\frac{\partial h}{\partial \bar{z}}(p) + \frac{\partial h}{\partial z}(p)\frac{\partial g}{\partial \bar{z}}(p)\Bigg)\Bigg),
\end{aligned}
\tag{5.3}
$$

where $K(G')(p)$ and $K(H')(p)$ are the HSCs at p with respect to G' and H', respectively. The choice of M' was such that $K(H')(p) = K(H,t)(p)$. Moreover, the decreasing property of HSC on submanifolds implies that $K\big(G+\lambda H, t\big)(p) \geq K\big(G'+\lambda H'\big)(p)$, where $K(G+\lambda H,t)(p)$ denotes the HSC at p with respect to $G+\lambda H$ in the direction of t. Therefore, (5.3) implies

$$
\begin{aligned}
&K\big(G + \lambda H, t\big)(p) \\
&\geq K\big(G' + \lambda H'\big)(p) \\
&= \frac{1}{\big(g(p)+\lambda h(p)\big)^3}\Bigg(g(p)^3 K(G')(p) + \lambda^2 h(p)^3 K(H,t)(p) + 2\lambda\Bigg(-h(p)\frac{\partial^2 g}{\partial z \partial \bar{z}}(p) \\
&\quad - g(p)\frac{\partial^2 h}{\partial z \partial \bar{z}}(p) + \frac{\partial g}{\partial z}(p)\frac{\partial h}{\partial \bar{z}}(p) + \frac{\partial h}{\partial z}(p)\frac{\partial g}{\partial \bar{z}}(p)\Bigg)\Bigg).
\end{aligned}
\tag{5.4}
$$

If λ is large enough, then the sign of the expression on the right hand side is determined by the sign of $K(H,t)(p)$, which is positive by assumption. Hence, the HSC at p with respect to $G + \lambda H$ is positive in the direction of t for a sufficiently large value of λ, say λ_t, i.e.,

$$K(G + \lambda_t H, t)(p) > 0.$$

It is clear from (5.4) that for any $\lambda > \lambda_t$, $K(G + \lambda H, t)(p)$ is still positive. We want a λ which works for all tangent vectors at p. Any tangent vector at p is a scalar multiple of some unit tangent vector at p (with respect to H). Therefore, it is enough to consider λ_t for $t \in S^1\big(T_pM\big) = \{t \in T_pM \ : \ H(t,t) = 1\}$. Since $S^1\big(T_pM\big)$ is compact, we conclude that there exists a λ which works for all t, i.e., $K(G+\lambda H, t)(p)$ is positive for any choice of tangent vector t at p. Thus, the HSC at p with respect to $G + \lambda H$ is positive in all the directions.

Remark 5.4. *The right hand side of the inequality* (5.4) *is* $O(\lambda^{-1})$*. Thus, the formula* (2.2) *of HSC implies that*

$$\frac{2\sum_{i,j,k,l=1}^n R_{i\bar{j}k\bar{l}}(p)\ \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l}{\sum_{i,j,k,l=1}^n \big(g_{i\bar{j}}(p) + \lambda h_{i\bar{j}}(p)\big)\big(g_{k\bar{l}}(p) + \lambda h_{k\bar{l}}(p)\big)\ \xi_i\bar{\xi}_j\xi_k\bar{\xi}_l} \geq O(\lambda^{-1}),$$

which further implies that

$$\sum_{i,j,k,l=1}^n R_{i\bar{j}k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l \geq O(\lambda)\sum_{i,j=1}^n \xi_i\bar{\xi}_i\xi_j\bar{\xi}_j. \tag{5.5}$$

5.3 Proof of Theorem 5.1

Suppose $\{G_t\}$ is a smooth family of Hermitian metrics with positive HSC on each fiber, and φ_t is the Hermitian form associated to the metric G_t. Fix a Hermitian metric $\widetilde{G}$ on X. For two vector fields Z_1 and Z_2 on X of type $(1,0)$ and $(0,1)$, respectively, we define a (1,1)-form $\widetilde{\Phi}$ at a point $p \in \pi^{-1}(t)$ as follows:

$$\widetilde{\Phi}(Z_1, Z_2)(p) \equiv \varphi_t\big(\mathrm{proj}_{\widetilde{G}}Z_1(p), \mathrm{proj}_{\widetilde{G}}Z_2(p)\big),$$

where $\mathrm{proj}_{\widetilde{G}}$ is the projection onto the fiber direction with respect to the metric $\widetilde{G}$. Then clearly $\widetilde{\Phi}$ is a C^∞, Hermitian (1,1)-form defined on X, and $\widetilde{\Phi}$ restricted to each fiber is equal to φ_t which is positive definite.

Suppose ω_Y is the associated (1,1)-form of the Hermitian metric on Y with positive HSC. Then for a sufficiently large value of μ, $\widetilde{\Phi} + \mu\pi^*(\omega_Y)$ is a positive definite Hermitian (1,1)-form defined on X. We fix one such μ_0, and consider the Hermitian metric on X with the associated (1,1)-form given by $\Phi = \widetilde{\Phi} + \mu_0\pi^*(\omega_Y)$. As mentioned in the definition of a Hermitian metric in Section 2.2, we shall refer to the associated (1,1)-form of a Hermitian metric as the Hermitian metric itself. We want to show that the Hermitian metric on X defined by $\Psi_\lambda = \Phi + \lambda\pi^*(\omega_Y)$ has positive HSC on X if λ is chosen large enough.

Let p be a point in X which lies in the fiber $X_0 = \pi^{-1}(t),\ t \in Y$. Since π is of maximal rank everywhere, locally there is a neighborhood U of p such that $U = W \times V$, where V is a neighborhood of $\pi(p)$ in Y, and W is a neighborhood of p in the fiber X_0. We may assume V and W are coordinate neighborhoods with local coordinates $(z_{s+1}, \ldots, z_n)$ in V and $(z_1, \ldots, z_s)$ in W. Then, $(z_1, \ldots, z_n)$ is a coordinate system around p in U. For computational purpose, we will choose $(z_{s+1}, \ldots, z_n)$ such that $\frac{\partial}{\partial z_{s+1}}, \ldots, \frac{\partial}{\partial z_n}$ are orthonormal at $\pi(p)$ with respect to the metric ω_Y. Let

$$\Phi = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\bar{j}}(z_1, \ldots, z_n)\, dz_i \wedge d\bar{z}_j,$$

and

$$\omega_Y = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta=s+1}^{n} \widetilde{g}_{\alpha\bar{\beta}}(z_{s+1}, \ldots, z_n)\, dz_\alpha \wedge d\bar{z}_\beta, \qquad \text{with } \widetilde{g}_{\alpha\bar{\beta}}(p) = \delta_{\alpha\bar{\beta}}.$$

Then the Hermitian metric Ψ_λ is given by

$$
\begin{aligned}
&\frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j \\
&= \frac{\sqrt{-1}}{2}\Bigg(\sum_{i,j=1}^{s} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j + \sum_{i=1}^{s}\sum_{\beta=s+1}^{n} g_{i\bar{\beta}}\, dz_i \wedge d\bar{z}_\beta + \sum_{\alpha=s+1}^{n}\sum_{j=1}^{s} g_{\alpha\bar{j}}\, dz_\alpha \wedge d\bar{z}_j \\
&\quad + \sum_{\alpha,\beta=s+1}^{n} \big(g_{\alpha\bar{\beta}} + (\lambda+\mu_0)\widetilde{g}_{\alpha\bar{\beta}}\big)\, dz_\alpha \wedge d\bar{z}_\beta\Bigg).
\end{aligned}
$$

Since we are considering large values of λ, and $\lambda+\mu_0$ is just another large constant, we can replace $\lambda+\mu_0$ with λ in the rest of the proof, i.e., the above expression changes to

$$
\begin{aligned}
&\frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j \\
&= \frac{\sqrt{-1}}{2}\Bigg(\sum_{i,j=1}^{s} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j + \sum_{i=1}^{s}\sum_{\beta=s+1}^{n} g_{i\bar{\beta}}\, dz_i \wedge d\bar{z}_\beta + \sum_{\alpha=s+1}^{n}\sum_{j=1}^{s} g_{\alpha\bar{j}}\, dz_\alpha \wedge d\bar{z}_j \\
&\quad + \sum_{\alpha,\beta=s+1}^{n} \big(g_{\alpha\bar{\beta}} + \lambda\widetilde{g}_{\alpha\bar{\beta}}\big)\, dz_\alpha \wedge d\bar{z}_\beta\Bigg).
\end{aligned}
$$

Let A be the $s\times s$ matrix with coefficients $g_{a\bar{b}}(p)$ for $1\le a,\, b\le s$, and A_{ab} be the $(a,b)^{th}$ cofactor of the matrix A. Then, using the following formula to compute the determinant of a block matrix:

$$
\det\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det(P)\,\det(S - RP^{-1}Q),
$$

we obtain the following expressions for inverse elements, with the assumption that

$1 \leq a, b \leq s$, and $s+1 \leq \chi, \eta \leq n$,

$$\begin{aligned} h^{a\bar{b}}(p) &= \frac{\lambda^{n-s} \det A_{ab} + O(\lambda^{n-s-1})}{\lambda^{n-s} \det A + O(\lambda^{n-s-1})}, \\ h^{\chi\bar{\chi}}(p) &= \frac{\lambda^{n-s-1} \det A + O(\lambda^{n-s-2})}{\lambda^{n-s} \det A + O(\lambda^{n-s-1})}, \\ h^{a\bar{\eta}}(p) = \ h^{\chi\bar{b}}(p) &= O(\lambda^{-1}), \qquad \text{and} \qquad h^{\chi\bar{\eta}}(p) = \ O(\lambda^{-2}), \qquad \chi \neq \eta. \end{aligned} \tag{5.6}$$

Now, we check the conditions of Lemma 5.2:

1. For $1 \leq i, j, k, l \leq s$, the formula (2.1) implies that the components of curvature tensor are given by

$$\begin{aligned} R_{i\bar{j}k\bar{l}}(p) = \ & -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \overline{z}_l}(p) + \sum_{a,b=1}^{s} h^{a\bar{b}}(p) \frac{\partial g_{i\bar{a}}}{\partial z_k}(p) \frac{\partial g_{b\bar{j}}}{\partial \overline{z}_l}(p) \\ & + \sum_{\sim(u,v \leq s)} h^{u\bar{v}}(p) \frac{\partial g_{i\bar{u}}}{\partial z_k}(p) \frac{\partial g_{v\bar{j}}}{\partial \overline{z}_l}(p). \end{aligned}$$

Dependence of the inverse elements on λ, as given in (5.6), reduces the above expression to

$$R_{i\bar{j}k\bar{l}}(p) = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \overline{z}_l}(p) + \sum_{a,b=1}^{s} h^{a\bar{b}}(p) \frac{\partial g_{i\bar{a}}}{\partial z_k}(p) \frac{\partial g_{b\bar{j}}}{\partial \overline{z}_l}(p) + O(\lambda^{-1}).$$

For large λ, we note that $\lim_{\lambda \to \infty} h^{a\bar{b}}(p) = \frac{\det A_{ab}}{\det A}$ is the inverse element of the $(a, b)^{th}$ coefficient of metric tensor associated to φ_t $(p \in \pi^{-1}(t))$. Therefore,

$$\begin{aligned} \lim_{\lambda \to \infty} R_{i\bar{j}k\bar{l}}(p) &= \lim_{\lambda \to \infty} \left(-\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \overline{z}_l}(p) + \sum_{a,b=1}^{s} h^{a\bar{b}}(p) \frac{\partial g_{i\bar{a}}}{\partial z_k}(p) \frac{\partial g_{b\bar{j}}}{\partial \overline{z}_l}(p) + O(\lambda^{-1}) \right) \\ &= -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \overline{z}_l}(p) + \sum_{a,b=1}^{s} \frac{\det A_{ab}}{\det A} \frac{\partial g_{i\bar{a}}}{\partial z_k}(p) \frac{\partial g_{b\bar{j}}}{\partial \overline{z}_l}(p). \end{aligned}$$

The expression

$$2\sum_{i,j,k,l=1}^{s}\left(-\frac{\partial^2 g_{i\bar{j}}}{\partial z_k\partial \bar{z}_l}(p)+\sum_{a,b=1}^{s}\frac{\det A_{ab}}{\det A}\frac{\partial g_{i\bar{a}}}{\partial z_k}(p)\frac{\partial g_{b\bar{j}}}{\partial \bar{z}_l}(p)\right)\xi_i\overline{\xi}_j\xi_k\overline{\xi}_l$$

is the numerator of the HSC of φ_t on the fiber $\pi^{-1}(t)$ which, by assumption, is bounded below by a positive constant. Thus, for a sufficiently large choice of λ, the expression

$$\begin{aligned}&\sum_{i,j,k,l=1}^{s} R_{i\bar{j}k\bar{l}}\xi_i\overline{\xi}_j\xi_k\overline{\xi}_l\\ =&\sum_{i,j,k,l=1}^{s}\left(-\frac{\partial^2 g_{i\bar{j}}}{\partial z_k\partial \bar{z}_l}(p)+\sum_{a,b=1}^{s}h^{a\bar{b}}(p)\frac{\partial g_{i\bar{a}}}{\partial z_k}(p)\frac{\partial g_{b\bar{j}}}{\partial \bar{z}_l}(p)+O(\lambda^{-1})\right)\xi_i\overline{\xi}_j\xi_k\overline{\xi}_l\end{aligned}$$

is bounded below by a positive constant. This proves that the first condition of Lemma 5.2 is satisfied for λ sufficiently large.

2. If $\min(i,j,k,l)\le s$, then the dependence of inverse elements on λ, as described in (5.6), and the fact that $\widetilde{g}_{\alpha\bar{\beta}}$'s are functions of $z_{s+1},\ldots,z_n$ only, imply that the formula (2.1) gives $|R_{i\bar{j}k\bar{l}}(p)|\le O(1)$ when $\min(i,j,k,l)\le s$. Therefore, the second condition of Lemma 5.2 is also satisfied when λ is large enough.

3. $s+1\le i,j,k,l\le n$:

   In this case, we consider a hypersurface M' around p defined by $\{z_1=\ldots=z_s=0\}$. Let G' be the induced metric of Φ on M', and $\widetilde{G}'$ be the induced metric of $\pi^*(\omega_Y)$ on M' so that

$$G'+\lambda\widetilde{G}'=\sum_{\alpha,\beta=s+1}^{n}\left(g_{\alpha\bar{\beta}}+\lambda\widetilde{g}_{\alpha\bar{\beta}}\right)dz_\alpha\otimes d\bar{z}_\beta$$

is the induced metric of Ψ_λ on M'. Clearly, $\widetilde{G}'$ has positive HSC at $p \in M'$. Lemma 5.3 implies that $G' + \lambda\widetilde{G}'$ also has positive HSC at p for a sufficiently large choice of λ. Therefore, the numerator of (2.2) is positive with respect to $G' + \lambda\widetilde{G}'$, i.e., if $R'_{\alpha\beta\gamma\delta}$ denote the components of the curvature tensor (for $\alpha, \beta, \gamma, \delta = s+1, \ldots, n$) with respect to the induced metric $G' + \lambda\widetilde{G}'$, then

$$\sum_{\alpha,\beta,\gamma,\delta=s+1}^{n} R'_{\alpha\bar{\beta}\gamma\bar{\delta}}(p)\xi_\alpha\overline{\xi}_\beta\xi_\gamma\overline{\xi}_\delta > 0.$$

The decreasing property of HSC on submanifolds implies that

$$\begin{aligned} K\Big(\Psi_\lambda, (0, \ldots, 0, \xi_{s+1}, \ldots, \xi_n)\Big)(p) \geq & K\Big(\Psi_\lambda|_{M'}, (\xi_{s+1}, \ldots, \xi_n)\Big)(p) \\ = & K\big(G' + \lambda\widetilde{G}', (\xi_{s+1}, \ldots, \xi_n)\big)(p), \end{aligned}$$

i.e.,

$$\begin{aligned} & \frac{2\sum_{\alpha,\beta,\gamma,\delta=s+1}^{n} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(p)\,\xi_\alpha\overline{\xi}_\beta\xi_\gamma\overline{\xi}_\delta}{\sum_{\alpha,\beta,\gamma,\delta=s+1}^{n}\big(g_{\alpha\bar{\beta}}(p) + \lambda\widetilde{g}_{\alpha\bar{\beta}}(p)\big)\big(g_{\gamma\bar{\delta}}(p) + \lambda\widetilde{g}_{\gamma\bar{\delta}}(p)\big)\,\xi_\alpha\overline{\xi}_\beta\xi_\gamma\overline{\xi}_\delta} \\ \geq & \frac{2\sum_{\alpha,\beta,\gamma,\delta=s+1}^{n} R'_{\alpha\bar{\beta}\gamma\bar{\delta}}(p)\,\xi_\alpha\overline{\xi}_\beta\xi_\gamma\overline{\xi}_\delta}{\sum_{\alpha,\beta,\gamma,\delta=s+1}^{n}\big(g_{\alpha\bar{\beta}}(p) + \lambda\widetilde{g}_{\alpha\bar{\beta}}(p)\big)\big(g_{\gamma\bar{\delta}}(p) + \lambda\widetilde{g}_{\gamma\bar{\delta}}(p)\big)\,\xi_\alpha\overline{\xi}_\beta\xi_\gamma\overline{\xi}_\delta}, \end{aligned}$$

which implies

$$\sum_{\alpha,\beta,\gamma,\delta=s+1}^{n} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(p)\,\xi_\alpha\overline{\xi}_\beta\xi_\gamma\overline{\xi}_\delta \geq \sum_{\alpha,\beta,\gamma,\delta=s+1}^{n} R'_{\alpha\bar{\beta}\gamma\bar{\delta}}(p)\,\xi_\alpha\overline{\xi}_\beta\xi_\gamma\overline{\xi}_\delta. \tag{5.7}$$

The expression on the right hand side of the above inequality is positive for λ sufficiently large. Therefore, the expression on the left hand side is also positive. This proves the third and last condition of Lemma 5.2 for λ sufficiently large.

According to (5.5) in Remark 5.4, the right hand side of the inequality (5.7), and consequently the left hand side of the inequality, satisfy a stronger positivity statement, namely

$$\sum_{\alpha,\beta,\gamma,\delta=s+1}^{n} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(p)\,\xi_\alpha\overline{\xi}_\beta\xi_\gamma\overline{\xi}_\delta \geq\ O(\lambda)\sum_{\alpha,\beta=s+1}^{n}\xi_\alpha\overline{\xi}_\alpha\xi_\beta\overline{\xi}_\beta.$$

Therefore, the inequality $K_2/K_1 \geq \mathcal{K}$ in the statement of Lemma 5.2 is satisfied for λ large enough, and hence, the lemma implies that Ψ_λ has positive HSC at p for λ sufficiently large.

Let $U_{p,\lambda}$ be a neighborhood of p in which the HSC with respect to Ψ_λ is positive everywhere, i.e., the HSC at every point $q \in U_{p,\lambda}$ is positive in all directions. Then [Wu73, Lemma 4] implies that the HSC at q in a direction $t \in T_qX$ (with respect to Ψ_λ) is equal to the Gaussian curvature at q with respect to the induced metric of Ψ_λ on a 1-dimensional submanifold tangent to t. Dependence of the Gaussian curvature on λ (in (5.4)) implies that if $K(\Psi_\lambda, t)(q) > 0$, then $K(\Psi_{\lambda'}, t)(q) > 0$ for any $\lambda' > \lambda$.

Therefore, using the compactness property of X, we conclude that there exists a sufficiently large value of λ such that the HSC of X with respect to Ψ_λ is positive everywhere.

Chapter 6

Counterexamples In The Semi-Definite HSC Case

This chapter shows, with the help of some examples, that variations of results proved in Chapter 5 might not be valid in the case of semi-definite HSC. In particular, the examples mentioned in this chapter are direct counterexamples to Lemma 5.2 in the respective semi-definite cases.

For all the examples in this chapter, we define $X = D_1 \times D_2$ to be the bi-disk and $Y = D_2$. Let (z_1, z_2) be the coordinate system in X. The holomorphic map $\pi : X \to Y$ is given by the projection onto the second coordinate, i.e., $(z_1, z_2) \mapsto z_2$. Clearly, π is of maximal rank everywhere. In Chapter 5, the warp factor λ (or s in Chapter 3) was chosen to be a constant. However, in this chapter, the warp factor λ will depend smoothly on the base space Y, whereby we show that the counterexamples cannot be ruled out simply by allowing λ to be non-constant.

We would like to remark here that the counterexamples given in this chapter only demonstrate the "non-existence of a warped metric" with semi-negative HSC.

Whether it is possible to obtain a metric of semi-definite HSC in the given setup, using some other method, is still an open question.

6.1 Negative HSC on the base and semi-negative HSC along the fibers

The Hermitian metric $\omega_Y = (1+z_2\overline{z}_2)\, dz_2 \otimes d\overline{z}_2$ on Y has negative HSC everywhere on Y, and the following tensor on X

$$\Phi = e^{-2z_2\overline{z}_2}\big(1 + (z_1\overline{z}_1)^2 e^{-4z_2\overline{z}_2}\big)\, dz_1 \otimes d\overline{z}_1$$

yields a Hermitian metric of semi-negative HSC when restricted to any of the fibers $\pi^{-1}(z_2)$, $z_2 \in Y$. Clearly, Φ varies smoothly with respect to the base points in Y. For an arbitrary smooth real-valued positive function $\lambda(z_2)$ on Y, we would like to show that the Hermitian metric

$$\begin{aligned} G &= \Phi + \lambda\pi^*(\omega_Y) \\ &= e^{-2z_2\overline{z}_2}\big(1 + (z_1\overline{z}_1)^2 e^{-4z_2\overline{z}_2}\big)\, dz_1 \otimes d\overline{z}_1 + \lambda(1 + z_2\overline{z}_2)\, dz_2 \otimes d\overline{z}_2 \end{aligned}$$

does not give semi-negative HSC everywhere on X.

The metric tensor associated to G is given by

$$\begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} e^{-2z_2\overline{z}_2}\big(1 + (z_1\overline{z}_1)^2 e^{-4z_2\overline{z}_2}\big) & 0 \\ 0 & \lambda(1 + z_2\overline{z}_2) \end{pmatrix},$$

where $g_{1\bar{1}} = e^{-2z_2\overline{z}_2}\big(1 + (z_1\overline{z}_1)^2 e^{-4z_2\overline{z}_2}\big)$, $g_{2\bar{2}} = \lambda(1 + z_2\overline{z}_2)$, $g_{1\bar{2}} = g_{2\bar{1}} = 0$, are smooth functions defined on X. The inverse of the above matrix is given by

$$\begin{pmatrix} g^{1\bar{1}} & g^{1\bar{2}} \\ g^{2\bar{1}} & g^{2\bar{2}} \end{pmatrix} = \begin{pmatrix} g_{1\bar{1}}^{-1} & 0 \\ 0 & g_{2\bar{2}}^{-1} \end{pmatrix} = \begin{pmatrix} \frac{e^{2z_2\bar{z}_2}}{1+(z_1\bar{z}_1)^2e^{-4z_2\bar{z}_2}} & 0 \\ 0 & \frac{1}{\lambda(1+z_2\bar{z}_2)} \end{pmatrix}.$$

All the derivatives of $g_{1\bar{2}}$ and $g_{2\bar{1}}$ are clearly zero. Moreover, since $g_{2\bar{2}}$ does not depend on z_1, the derivatives of $g_{2\bar{2}}$ with respect to z_1 and $\bar{z}_1$ are also zero. Therefore, following are the only non-zero derivatives of $g_{i\bar{j}}$, $i, j = 1, 2$:

$$\begin{aligned}
\frac{\partial g_{1\bar{1}}}{\partial z_1} &= 2z_1\bar{z}_1^2e^{-6z_2\bar{z}_2}, \qquad & \frac{\partial g_{1\bar{1}}}{\partial z_2} &= -2\bar{z}_2e^{-2z_2\bar{z}_2}\Big(1+3(z_1\bar{z}_1)^2e^{-4z_2\bar{z}_2}\Big), \\
\frac{\partial g_{1\bar{1}}}{\partial \bar{z}_1} &= 2z_1^2\bar{z}_1e^{-6z_2\bar{z}_2}, \qquad & \frac{\partial g_{1\bar{1}}}{\partial \bar{z}_2} &= -2z_2e^{-2z_2\bar{z}_2}\Big(1+3(z_1\bar{z}_1)^2e^{-4z_2\bar{z}_2}\Big),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 g_{1\bar{1}}}{\partial z_1\partial \bar{z}_1} &= 4z_1\bar{z}_1e^{-6z_2\bar{z}_2}, \\
\frac{\partial^2 g_{1\bar{1}}}{\partial z_1\partial \bar{z}_2} &= -12z_1\bar{z}_1^2z_2e^{-6z_2\bar{z}_2}, \\
\frac{\partial^2 g_{1\bar{1}}}{\partial z_2\partial \bar{z}_1} &= -12z_1^2\bar{z}_1\bar{z}_2e^{-6z_2\bar{z}_2}, \\
\frac{\partial^2 g_{1\bar{1}}}{\partial z_2\partial \bar{z}_2} &= -2e^{-2z_2\bar{z}_2}\Big(1-2z_2\bar{z}_2+3(1-6z_2\bar{z}_2)(z_1\bar{z}_1)^2e^{-4z_2\bar{z}_2}\Big),
\end{aligned}$$

$$\frac{\partial g_{2\bar{2}}}{\partial z_2} = \lambda\bar{z}_2+(1+z_2\bar{z}_2)\frac{\partial\lambda}{\partial z_2}, \qquad \frac{\partial g_{2\bar{2}}}{\partial \bar{z}_2} = \lambda z_2+(1+z_2\bar{z}_2)\frac{\partial\lambda}{\partial \bar{z}_2},$$

$$\frac{\partial^2 g_{2\bar{2}}}{\partial z_2\partial \bar{z}_2} = \lambda+z_2\frac{\partial\lambda}{\partial z_2}+\bar{z}_2\frac{\partial\lambda}{\partial \bar{z}_2}+(1+z_2\bar{z}_2)\frac{\partial^2\lambda}{\partial z_2\partial \bar{z}_2}.$$

Using the formula (2.1), the components of the curvature tensor with respect to

G are obtained as follows:

$$\begin{aligned}
R_{1\bar{1}1\bar{1}} &= -\frac{4z_1\overline{z}_1e^{-6z_2\overline{z}_2}}{1+(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}},\\
R_{1\bar{1}1\bar{2}} &= \frac{8z_1\overline{z}_1^2z_2e^{-6z_2\overline{z}_2}}{1+(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}},\\
R_{1\bar{1}2\bar{1}} &= \frac{8z_1^2\overline{z}_1\overline{z}_2e^{-6z_2\overline{z}_2}}{1+(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}},\\
R_{1\bar{1}2\bar{2}} &= \frac{2e^{-2z_2\overline{z}_2}\Big(1+\big(4+3(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}-8z_2\overline{z}_2\big)(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}\Big)}{1+(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}},
\end{aligned}$$

and

$$R_{2\bar{2}2\bar{2}} = -\frac{\lambda}{1+z_2\overline{z}_2}+(1+z_2\overline{z}_2)\bigg(-\frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial z_2}\frac{\partial\lambda}{\partial\overline{z}_2}\bigg).$$

Using (2.2), the HSC of X at a point (z_1, z_2) with respect to G in the direction of a unit tangent vector $\xi = (\xi_1, \xi_2)$ is given by

$$\begin{aligned}
K(\xi) &= R_{1\bar{1}1\bar{1}}\ \xi_1\overline{\xi}_1\xi_1\overline{\xi}_1 + R_{1\bar{1}1\bar{2}}\ \xi_1\overline{\xi}_1\xi_1\overline{\xi}_2 + R_{1\bar{1}2\bar{1}}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_1\\
&\quad + R_{1\bar{1}2\bar{2}}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_2 + R_{2\bar{2}2\bar{2}}\ \xi_2\overline{\xi}_2\xi_2\overline{\xi}_2\\
&= -\frac{4z_1\overline{z}_1e^{-6z_2\overline{z}_2}}{1+(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}}\ \xi_1\overline{\xi}_1\xi_1\overline{\xi}_1 + \frac{8z_1\overline{z}_1^2z_2e^{-6z_2\overline{z}_2}}{1+(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}}\ \xi_1\overline{\xi}_1\xi_1\overline{\xi}_2\\
&\quad + \frac{8z_1^2\overline{z}_1\overline{z}_2e^{-6z_2\overline{z}_2}}{1+(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_1\\
&\quad + \frac{2e^{-2z_2\overline{z}_2}\Big(1+\big(4+3(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}-8z_2\overline{z}_2\big)(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}\Big)}{1+(z_1\overline{z}_1)^2e^{-4z_2\overline{z}_2}}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_2\\
&\quad - \Bigg(\frac{\lambda}{1+z_2\overline{z}_2}-(1+z_2\overline{z}_2)\bigg(-\frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial z_2}\frac{\partial\lambda}{\partial\overline{z}_2}\bigg)\Bigg)\ \xi_2\overline{\xi}_2\xi_2\overline{\xi}_2.
\end{aligned}$$

Now, we shall compute $K(\xi)$ at the point $(0,0) \in X$. A unit tangent vector (ξ_1, ξ_2) at (0,0) satisfies

$$g_{1\bar{1}}(0,0)\xi_1\overline{\xi}_1 + g_{2\bar{2}}(0,0)\xi_2\overline{\xi}_2 = 1$$

$$\Rightarrow \xi_1\overline{\xi}_1 + \lambda\xi_2\overline{\xi}_2 = 1$$

$$\Rightarrow \xi_1\overline{\xi}_1 = 1 - \lambda\xi_2\overline{\xi}_2.$$

Using the above relation between $\xi_1\overline{\xi}_1$ and $\xi_2\overline{\xi}_2$, we obtain the HSC at the origin as follows:

$$\begin{aligned}
K_{(0,0)}(\xi) &= 2\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_2 - \left(\lambda(0) + \frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}(0) - \frac{1}{\lambda(0)}\frac{\partial\lambda}{\partial z_2}(0)\frac{\partial\lambda}{\partial\overline{z}_2}(0)\right)\ \xi_2\overline{\xi}_2\xi_2\overline{\xi}_2 \\
&= \left(2\ \xi_1\overline{\xi}_1 - \left(\lambda(0) + \frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}(0) - \frac{1}{\lambda(0)}\frac{\partial\lambda}{\partial z_2}(0)\frac{\partial\lambda}{\partial\overline{z}_2}(0)\right)\ \xi_2\overline{\xi}_2\right)\ \xi_2\overline{\xi}_2 \\
&= \left(2 - \left(3\lambda(0) + \frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}(0) - \frac{1}{\lambda(0)}\frac{\partial\lambda}{\partial z_2}(0)\frac{\partial\lambda}{\partial\overline{z}_2}(0)\right)\ \xi_2\overline{\xi}_2\right)\ \xi_2\overline{\xi}_2.
\end{aligned}$$

For any value of λ and its derivatives at $z_2 = 0$, there exists a non-zero ξ_2 with $\xi_2\overline{\xi}_2$ small enough such that

$$\xi_2\overline{\xi}_2 < 2\left(3\lambda(0) + \frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}(0) - \frac{1}{\lambda(0)}\frac{\partial\lambda}{\partial z_2}(0)\frac{\partial\lambda}{\partial\overline{z}_2}(0)\right)^{-1},$$

i.e.,

$$2 - \left(3\lambda(0) + \frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}(0) - \frac{1}{\lambda(0)}\frac{\partial\lambda}{\partial z_2}(0)\frac{\partial\lambda}{\partial\overline{z}_2}(0)\right)\ \xi_2\overline{\xi}_2 > 0.$$

Therefore, for any value of λ and its derivatives at $z_2 = 0$, there exists a unit tangent vector $\xi = (\xi_1, \xi_2)$ such that the HSC at (0,0) with respect to G in the direction of ξ is positive.

6.2 Positive HSC on the base and semi-positive HSC along the fibers

The Hermitian metric $\omega_Y = \frac{1}{1+z_2\bar{z}_2}\, dz_2 \otimes d\bar{z}_2$ on Y has positive HSC everywhere on Y, and the following tensor on X

$$\Phi = \frac{e^{2z_2\bar{z}_2}}{1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}}\, dz_1 \otimes d\bar{z}_1$$

yields a Hermitian metric of semi-positive HSC when restricted to any of the fibers $\pi^{-1}(z_2)$, $z_2 \in Y$. Clearly, Φ varies smoothly with respect to the base points in Y. For an arbitrary smooth real-valued positive function $\lambda(z_2)$ on Y, we would like to show that the Hermitian metric

$$\begin{aligned} G &= \Phi + \lambda\pi^*(\omega_Y) \\ &= \frac{e^{2z_2\bar{z}_2}}{1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}}\, dz_1 \otimes d\bar{z}_1 + \frac{\lambda}{1+z_2\bar{z}_2}\, dz_2 \otimes d\bar{z}_2 \end{aligned}$$

does not give semi-positive HSC everywhere on X.

The metric tensor associated to G is given by

$$\begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \frac{e^{2z_2\bar{z}_2}}{1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}} & 0 \\ 0 & \frac{\lambda}{1+z_2\bar{z}_2} \end{pmatrix},$$

where $g_{1\bar{1}} = \frac{e^{2z_2\bar{z}_2}}{1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}}$, $g_{2\bar{2}} = \frac{\lambda}{1+z_2\bar{z}_2}$, $g_{1\bar{2}} = g_{2\bar{1}} = 0$, are smooth functions defined on X. The inverse of the above matrix is given by

$$\begin{pmatrix} g^{1\bar{1}} & g^{1\bar{2}} \\ g^{2\bar{1}} & g^{2\bar{2}} \end{pmatrix} = \begin{pmatrix} g_{1\bar{1}}^{-1} & 0 \\ 0 & g_{2\bar{2}}^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}}{e^{2z_2\bar{z}_2}} & 0 \\ 0 & \frac{1+z_2\bar{z}_2}{\lambda} \end{pmatrix}.$$

As in the previous section, the following are the only non-zero derivatives of $g_{i\bar{j}}$, $i,j = 1,2$:

$$\begin{aligned}
\frac{\partial g_{1\bar{1}}}{\partial z_1} &= -\frac{2z_1\bar{z}_1^2 e^{6z_2\bar{z}_2}}{\left(1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)^2}, \\
\frac{\partial g_{1\bar{1}}}{\partial \bar{z}_1} &= -\frac{2z_1^2\bar{z}_1 e^{6z_2\bar{z}_2}}{\left(1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)^2}, \\
\frac{\partial g_{1\bar{1}}}{\partial z_2} &= \frac{2\bar{z}_2 e^{2z_2\bar{z}_2}\left(1-(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)}{\left(1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)^2}, \\
\frac{\partial g_{1\bar{1}}}{\partial \bar{z}_2} &= \frac{2z_2 e^{2z_2\bar{z}_2}\left(1-(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)}{\left(1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)^2}, \\
\frac{\partial^2 g_{1\bar{1}}}{\partial z_1\partial \bar{z}_1} &= -\frac{4z_1\bar{z}_1 e^{6z_2\bar{z}_2}\left(1-(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)}{\left(1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)^3}, \\
\frac{\partial^2 g_{1\bar{1}}}{\partial z_1\partial \bar{z}_2} &= -\frac{4z_1\bar{z}_1^2 z_2 e^{6z_2\bar{z}_2}\left(3-(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)}{\left(1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)^3}, \\
\frac{\partial^2 g_{1\bar{1}}}{\partial z_2\partial \bar{z}_1} &= -\frac{4z_1^2\bar{z}_1 \bar{z}_2 e^{6z_2\bar{z}_2}\left(3-(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)}{\left(1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)^3}, \\
\frac{\partial^2 g_{1\bar{1}}}{\partial z_2\partial \bar{z}_2} &= \frac{2e^{2z_2\bar{z}_2}\Big(1+2z_2\bar{z}_2\big(1-6(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\big)-\big(1-2z_2\bar{z}_2\big)(z_1\bar{z}_1)^4 e^{8z_2\bar{z}_2}\Big)}{\left(1+(z_1\bar{z}_1)^2 e^{4z_2\bar{z}_2}\right)^3},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g_{2\bar{2}}}{\partial z_2} &= -\frac{\lambda\bar{z}_2}{(1+z_2\bar{z}_2)^2} + \frac{1}{1+z_2\bar{z}_2}\frac{\partial\lambda}{\partial z_2}, \\
\frac{\partial g_{2\bar{2}}}{\partial \bar{z}_2} &= -\frac{\lambda z_2}{(1+z_2\bar{z}_2)^2} + \frac{1}{1+z_2\bar{z}_2}\frac{\partial\lambda}{\partial \bar{z}_2}, \\
\frac{\partial^2 g_{2\bar{2}}}{\partial z_2\partial \bar{z}_2} &= -\frac{(1-z_2\bar{z}_2)\lambda}{(1+z_2\bar{z}_2)^3} - \frac{z_2}{(1+z_2\bar{z}_2)^2}\frac{\partial\lambda}{\partial z_2} - \frac{\bar{z}_2}{(1+z_2\bar{z}_2)^2}\frac{\partial\lambda}{\partial \bar{z}_2} + \frac{1}{1+z_2\bar{z}_2}\frac{\partial^2\lambda}{\partial z_2\partial \bar{z}_2}.
\end{aligned}$$

The components of the curvature tensor with respect to G are obtained as follows:

$$\begin{aligned}
R_{1\bar{1}1\bar{1}} &= \frac{4z_1\overline{z}_1 e^{6z_2\overline{z}_2}}{\left(1+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\right)^3},\\
R_{1\bar{1}1\bar{2}} &= \frac{8z_1\overline{z}_1^2 z_2 e^{6z_2\overline{z}_2}}{\left(1+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\right)^3},\\
R_{1\bar{1}2\bar{1}} &= \frac{8z_1^2\overline{z}_1\overline{z}_2 e^{6z_2\overline{z}_2}}{\left(1+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\right)^3},\\
R_{1\bar{1}2\bar{2}} &= -\frac{2e^{2z_2\overline{z}_2}\Big(1-\big(8z_2\overline{z}_2+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\big)(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\Big)}{\left(1+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\right)^3},\\
R_{2\bar{2}2\bar{2}} &= \frac{\lambda}{(1+z_2\overline{z}_2)^3}+\frac{1}{1+z_2\overline{z}_2}\left(-\frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial z_2}\frac{\partial\lambda}{\partial\overline{z}_2}\right).
\end{aligned}$$

The HSC of X at a point (z_1, z_2) with respect to G in the direction of a unit tangent vector $\xi = (\xi_1, \xi_2)$ is given by

$$\begin{aligned}
K(\xi) &= R_{1\bar{1}1\bar{1}}\ \xi_1\overline{\xi}_1\xi_1\overline{\xi}_1 + R_{1\bar{1}1\bar{2}}\ \xi_1\overline{\xi}_1\xi_1\overline{\xi}_2 + R_{1\bar{1}2\bar{1}}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_1\\
&\quad + R_{1\bar{1}2\bar{2}}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_2 + R_{2\bar{2}2\bar{2}}\ \xi_2\overline{\xi}_2\xi_2\overline{\xi}_2\\
&= \frac{4z_1\overline{z}_1 e^{6z_2\overline{z}_2}}{\left(1+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\right)^3}\ \xi_1\overline{\xi}_1\xi_1\overline{\xi}_1 + \frac{8z_1\overline{z}_1^2 z_2 e^{6z_2\overline{z}_2}}{\left(1+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\right)^3}\ \xi_1\overline{\xi}_1\xi_1\overline{\xi}_2\\
&\quad + \frac{8z_1^2\overline{z}_1\overline{z}_2 e^{6z_2\overline{z}_2}}{\left(1+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\right)^3}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_1\\
&\quad - \frac{2e^{2z_2\overline{z}_2}\Big(1-\big(8z_2\overline{z}_2+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\big)(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\Big)}{\left(1+(z_1\overline{z}_1)^2 e^{4z_2\overline{z}_2}\right)^3}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_2\\
&\quad + \left(\frac{\lambda}{(1+z_2\overline{z}_2)^3}+\frac{1}{1+z_2\overline{z}_2}\left(-\frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial z_2}\frac{\partial\lambda}{\partial\overline{z}_2}\right)\right)\ \xi_2\overline{\xi}_2\xi_2\overline{\xi}_2.
\end{aligned}$$

A unit tangent vector (ξ_1, ξ_2) at (0,0) satisfies

$$\xi_1\overline{\xi}_1 = 1-\lambda\xi_2\overline{\xi}_2,$$

just like in Section 6.1. Then the HSC at the origin is given by

$$\begin{aligned} K_{(0,0)}(\xi) = & -2\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_2 + \left(\lambda(0) - \frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}(0) + \frac{1}{\lambda(0)}\frac{\partial\lambda}{\partial z_2}(0)\frac{\partial\lambda}{\partial\overline{z}_2}(0)\right)\ \xi_2\overline{\xi}_2\xi_2\overline{\xi}_2 \\ = & -\left(2\ \xi_1\overline{\xi}_1 - \left(\lambda(0) - \frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}(0) + \frac{1}{\lambda(0)}\frac{\partial\lambda}{\partial z_2}(0)\frac{\partial\lambda}{\partial\overline{z}_2}(0)\right)\ \xi_2\overline{\xi}_2\right)\ \xi_2\overline{\xi}_2 \\ = & -\left(2 - \left(3\lambda(0) - \frac{\partial^2\lambda}{\partial z_2\partial\overline{z}_2}(0) + \frac{1}{\lambda(0)}\frac{\partial\lambda}{\partial z_2}(0)\frac{\partial\lambda}{\partial\overline{z}_2}(0)\right)\ \xi_2\overline{\xi}_2\right)\ \xi_2\overline{\xi}_2. \end{aligned}$$

A similar argument as in Section 6.1 implies that for any value of λ and its derivatives at $z_2 = 0$, we can find a unit tangent vector $\xi = (\xi_1, \xi_2)$ at the origin such that $K_{(0,0)}(\xi)$ is negative.

6.3 Zero HSC on the base and fibers

The base space Y has the flat metric $\omega_Y = dz_2 \otimes d\overline{z}_2$, and the following tensor on X

$$\Phi = \ e^{2z_2\overline{z}_2}\ dz_1 \otimes d\overline{z}_1$$

yields a Hermitian metric of zero HSC when restricted to any of the fibers $\pi^{-1}(z_2)$, $z_2 \in Y$. Clearly, Φ varies smoothly with respect to the base points in Y. For an arbitrary smooth real-valued positive function $\lambda(z_2)$ on Y, we would like to show that the Hermitian metric

$$\begin{aligned} G &= \Phi + \lambda\pi^*(\omega_Y) \\ &= e^{2z_2\overline{z}_2}\ dz_1 \otimes d\overline{z}_1 + \lambda\ dz_2 \otimes d\overline{z}_2 \end{aligned}$$

does not give zero HSC everywhere on X.

The metric tensor associated to G is given by

$$\begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} e^{2z_2\overline{z}_2} & 0 \\ 0 & \lambda \end{pmatrix},$$

where $g_{1\bar{1}} = e^{2z_2\overline{z}_2}$, $g_{2\bar{2}} = \lambda$, $g_{1\bar{2}} = g_{2\bar{1}} = 0$, are smooth functions defined on X. The inverse of the above matrix is given by

$$\begin{pmatrix} g^{1\bar{1}} & g^{1\bar{2}} \\ g^{2\bar{1}} & g^{2\bar{2}} \end{pmatrix} = \begin{pmatrix} g_{1\bar{1}}^{-1} & 0 \\ 0 & g_{2\bar{2}}^{-1} \end{pmatrix} = \begin{pmatrix} e^{-2z_2\overline{z}_2} & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}.$$

The only non-zero derivatives in this case are the derivatives of $g_{1\bar{1}}$ and $g_{2\bar{2}}$ with respect to z_2 and $\overline{z}_2$, because both $g_{1\bar{1}}$ and $g_{2\bar{2}}$ depend only on z_2. These derivatives are given as follows:

$$\frac{\partial g_{1\bar{1}}}{\partial z_2} = 2\overline{z}_2 e^{2z_2\overline{z}_2}, \qquad \frac{\partial g_{1\bar{1}}}{\partial \overline{z}_2} = 2z_2 e^{2z_2\overline{z}_2}, \qquad \frac{\partial^2 g_{1\bar{1}}}{\partial z_2 \partial \overline{z}_2} = 2e^{2z_2\overline{z}_2} + 4z_2\overline{z}_2 e^{2z_2\overline{z}_2},$$

$$\frac{\partial g_{2\bar{2}}}{\partial z_2} = \frac{\partial \lambda}{\partial z_2}, \qquad \frac{\partial g_{2\bar{2}}}{\partial \overline{z}_2} = \frac{\partial \lambda}{\partial \overline{z}_2}, \qquad \frac{\partial^2 g_{2\bar{2}}}{\partial z_2 \partial \overline{z}_2} = -\frac{\partial^2 \lambda}{\partial z_2 \partial \overline{z}_2}.$$

The following are the only non-zero components of curvature tensor with respect to G:

$$R_{1\bar{1}2\bar{2}} = -2e^{2z_2\overline{z}_2},$$

$$R_{2\bar{2}2\bar{2}} = -\frac{\partial^2 \lambda}{\partial z_2 \partial \overline{z}_2} + \frac{1}{\lambda}\frac{\partial \lambda}{\partial z_2}\frac{\partial \lambda}{\partial \overline{z}_2}.$$

The HSC of X at a point (z_1, z_2) with respect to G in the direction of a unit tangent vector $\xi = (\xi_1, \xi_2)$ is given by

$$\begin{aligned} K(\xi) &= R_{1\bar{1}2\bar{2}}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_2 + R_{2\bar{2}2\bar{2}}\ \xi_2\overline{\xi}_2\xi_2\overline{\xi}_2 \\ &= -2e^{2z_2\overline{z}_2}\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_2 + \left(- \frac{\partial^2\lambda}{\partial z_2 \partial \overline{z}_2} + \frac{1}{\lambda}\frac{\partial\lambda}{\partial z_2}\frac{\partial\lambda}{\partial \overline{z}_2}\right)\ \xi_2\overline{\xi}_2\xi_2\overline{\xi}_2. \end{aligned}$$

As in previous sections, a unit tangent vector (ξ_1, ξ_2) at (0,0) satisfies

$$\xi_1\overline{\xi}_1 = 1 - \lambda\xi_2\overline{\xi}_2.$$

Therefore, the HSC at the origin is given by

$$\begin{aligned} K_{(0,0)}(\xi) &= -2\ \xi_1\overline{\xi}_1\xi_2\overline{\xi}_2 + \left(- \frac{\partial^2\lambda}{\partial z_2 \partial \overline{z}_2} + \frac{1}{\lambda}\frac{\partial\lambda}{\partial z_2}\frac{\partial\lambda}{\partial \overline{z}_2}\right)\ \xi_2\overline{\xi}_2\xi_2\overline{\xi}_2 \\ &= \left(-2 + \left(2\lambda - \frac{\partial^2\lambda}{\partial z_2 \partial \overline{z}_2} + \frac{1}{\lambda}\frac{\partial\lambda}{\partial z_2}\frac{\partial\lambda}{\partial \overline{z}_2}\right)\ \xi_2\overline{\xi}_2\right)\ \xi_2\overline{\xi}_2, \end{aligned}$$

which is clearly not zero for all possible values of ξ_2. In particular, any value of ξ_2 such that

$$\xi_2\overline{\xi}_2 < 2\left(2\lambda - \frac{\partial^2\lambda}{\partial z_2 \partial \overline{z}_2} + \frac{1}{\lambda}\frac{\partial\lambda}{\partial z_2}\frac{\partial\lambda}{\partial \overline{z}_2}\right)^{-1},$$

implies that $K_{(0,0)}(\xi)$ is negative.

Bibliography

[ACH15] Angelynn Alvarez, Ananya Chaturvedi, and Gordon Heier. Optimal pinching for the holomorphic sectional curvature of Hitchin's metrics on Hirzebruch surfaces. In *Rational points, rational curves, and entire holomorphic curves on projective varieties*, volume 654 of *Contemp. Math.*, pages 133–142. Amer. Math. Soc., Providence, RI, 2015.

[AHZ16] Angelynn Alvarez, Gordon Heier, and Fangyang Zheng. On projectivized vector bundles and positive holomorphic sectional curvature. arXiv:1606.08347, 2016.

[Ber66] Marcel Berger. Sur les variétés d'Einstein compactes. In *Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (Namur, 1965)*, pages 35–55. Librairie Universitaire, Louvain, 1966.

[BG65] Richard L. Bishop and Samuel I. Goldberg. On the topology of positively curved Kaehler manifolds. II. *Tôhoku Math. J. (2)*, 17:310–318, 1965.

[BO69] Richard L. Bishop and Barrett O'Neill. Manifolds of negative curvature. *Trans. Amer. Math. Soc.*, 145:1–49, 1969.

[Che77] Bang-yen Chen. Extrinsic spheres in Kähler manifolds. II. *Michigan Math. J.*, 24(1):97–102, 1977.

[Che89] Chi-Keung Cheung. Hermitian metrics of negative holomorphic sectional curvature on some hyperbolic manifolds. *Math. Z.*, 201(1):105–119, 1989.

[DT16] Simone Diverio and Stefano Trapani. Quasi-negative holomorphic sectional curvature and positivity of the canonical bundle. arXiv:1606.01381, 2016.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[Hit75] Nigel Hitchin. On the curvature of rational surfaces. In *Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973)*, pages 65–80. Amer. Math. Soc., Providence, R. I., 1975.

[HLW10] Gordon Heier, Steven S. Y. Lu, and Bun Wong. On the canonical line bundle and negative holomorphic sectional curvature. *Math. Res. Lett.*, 17(6):1101–1110, 2010.

[HM98] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. II*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1969 original, A Wiley-Interscience Publication.

[TY16] Valentino Tosatti and Xiaokui Yang. An extension of a theorem of Wu-Yau. arXiv:1506.01145, 2016.

[Wu73] Hung-Hsi Wu. A remark on holomorphic sectional curvature. *Indiana Univ. Math. J.*, 22:1103–1108, 1973.

[WY16a] Damin Wu and Shing-Tung Yau. Negative holomorphic curvature and positive canonical bundle. *Invent. Math.*, 204(2):595–604, 2016.

[WY16b] Damin Wu and Shing-Tung Yau. A remark on our paper "Negative holomorphic curvature and positive canonical bundle". arXiv:1609.01377, 2016.