ON THE POSITIVE HOLOMORPHIC SECTIONAL CURVATURE OF PROJECTIVIZED VECTOR BUNDLES OVER COMPACT COMPLEX MANIFOLDS

A Dissertation Presented to

the Faculty of the Department of Mathematics

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Angelynn Reario Alvarez

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Abstract

In complex geometry, there are few known examples of, and few known results about, manifolds with metrics of positive curvature. For instance, the geometry of fiber bundles and total spaces of fibrations over positively-curved complex manifolds is mysterious and not well-understood. In this dissertation, we study the existence of (pinched) metrics of positive curvature on a particular type of fiber bundle—namely metrics of positive holomorphic sectional curvature on projectivized vector bundles over compact complex manifolds. We first prove a general theorem for projectivized vector bundles, then we discuss a curvature pinching result for projectivized rank 2 vector bundles over complex projective space of dimension 1.

Contents

Chapter 1

Introduction

In the world of complex geometry, an often studied phenomenon is the dichotomy between manifolds of positive curvature and manifolds of negative curvature. In the positive case, few examples are known of manifolds with metrics of positive curvature. Additionally, there tend to be fewer known results about positively-curved manifolds compared with the corresponding situation in negative curvature. This disparity is due to the many difficulties which arise when dealing with positive curvature.

This dissertation primarily concentrates on the *holomorphic sectional curvature* of compact complex manifolds. The holomorphic sectional curvature of a Kähler manifold is precisely the Riemannian sectional curvature of the holomorphic planes in the tangent space of the manifold. We focus on this particular curvature because it has significant relationships to various notions in algebraic geometry which help in further studying the manifold structure. For instance, a result in [HW15] shows that projective manifolds which admit a Kähler metric of positive holomorphic sectional curvature are rationally connected. The holomorphic sectional curvature of a complex manifold also plays a role in determining its Kodaira dimension. For example, the relationship between semi-positive holomorphic sectional curvature and the Kodaira dimension of compact Hermitian manifolds is discussed in [Yan15]. Furthermore, results on the Kodaira dimension of projective manifolds of semi-negative holomorphic sectional curvature are discussed in [HLW15]. The holomorphic sectional curvature of a complex manifold can also determine the positivity of the canonical bundle (e.g., whether the canonical bundle is ample, numerically effective (nef), etc.). For instance, it was shown that a projective manifold which admits a Kähler metric of semi-negative holomorphic sectional curvature contains no rational curves and has nef canonical bundle (see [HLW15] and [Shi71]). In addition, the relationship between negative holomorphic sectional curvature and the ampleness of the canonical bundle of a projective manifold is discussed in [HLW10] and [WY15]. An extension of the result in [WY15] can be found in [TY15], which states that a compact Kähler manifold with negative holomorphic sectional curvature has ample canonical bundle. More relationships between curvature and the positivity of the canonical bundle and anti-canonical bundle are discussed in detail in Section 2.4.

In general, there are few examples known of compact complex manifolds which carry a Hermitian metric of *positively*-pinched holomorphic sectional curvature. A notable exception form the irreducible Hermitian symmetric spaces of compact type, whose pinching constants for the holomorphic sectional curvature can be found in [Che77, Table I]. Moreover, many difficulties arise when dealing with *positive* holomorphic sectional curvature. For example, we have the Curvature Decreasing Property of Subbundles (found in [Gri69], [Kob70], and [Wu73]), which effectively states that any complex submanifold of a Hermitian manifold of negative holomorphic sectional curvature will also have negative holomorphic sectional curvature. On the other hand, a complex submanifold of a Hermitian manifold of positive holomorphic sectional curvature is not guaranteed to also have positive holomorphic sectional curvature.

Because of these difficulties, it is a worthwhile endeavor to find and investigate metrics of positive curvature on complex manifolds. In this dissertation, we present several results on the existence of (pinched) metrics of positive holomorphic sectional curvature on total spaces of certain fibrations $\pi : P \to M$, namely where P is a projectivized vector bundle and M is a compact complex manifold of positive holomorphic sectional curvature. We also discuss an explicit curvature pinching constant for projectivized rank 2 vector bundles over $\mathbb{CP}^n$ (formally known as the Hirzebruch surfaces).

The work in this dissertation was motivated by several known results and open questions in the realm of positive curvature. In particular, this work was partially inspired by the following result proven by Cheung in *negative* curvature:

Theorem 1.1 ([Che89])**.** *Let $\pi : X \to Y$ be a holomorphic map of a compact complex manifold X into a complex manifold Y which has a Hermitian metric of negative holomorphic sectional curvature. Assume that π is of maximal rank everywhere and there exists a smooth family of Hermitian metrics on the fibers, which all have negative holomorphic sectional curvature. There there exists a Hermitian metric on X with negative holomorphic sectional curvature everywhere.*

Despite this result, the curvature and geometry of fiber bundles and fibrations are still mysterious and not well-understood in the positive case. A natural question to ask is: *Does the result of Cheung still hold true for metrics of positive holomorphic sectional curvature?* Arriving at an answer seems to be more involved than in the negative case—for instance, due to the Curvature Decreasing Property of Subbundles not being applicable. Hence, the investigation of fibrations and fiber bundles are left for a later occasion. As a primary stepping stone, we first consider the case of projectivized vector bundles on compact complex manifolds. The idea to projectivize vector bundles was prompted by a result proven by Hitchin in [Hit75] which states that the Hirzebruch surfaces admit a Kähler metric of positive holomorphic sectional curvature. Despite proving positivity, Hitchin's result did not yield any curvature pinching constants. Additionally, in [SY10], Yau posed the following open question from his list of open problems in Riemannian geometry: *Do all vector bundles over a manifold with positive [Riemannian] sectional curvature admit a complete metric with nonnegative [Riemannian] sectional curvature?* When we transplant Yau's question to the complex projective setting, "nonnegative curvature" naturally gets replaced by "positive curvature". The layout of this dissertation is as follows:

In Chapter 2, we review relevant definitions and topics from complex geometry, differential geometry, and algebraic geometry which are necessary for this dissertation.

In Chapter 3, we prove our main theorem on metrics of positive holomorphic sectional curvature for general projectivized rank $k \in \mathbb{N}$ vector bundles over compact Kähler manifolds, where the base manifold also has positive holomorphic sectional curvature. This theorem serves as a generalization of Hitchin's theorem in [Hit75].

Theorem 1.2 ([AHZ15])**.** *Let M be an n-dimensional compact Kähler manifold. Let E be holomorphic vector bundle over M and let $\pi : P = \mathbb{P}(E) \to M$ be the projectivization of E. If M has positive holomorphic sectional curvature, then P admits a Kähler metric with positive holomorphic sectional curvature.*

The proof requires the clever use of normal coordinates in which to do the curvature computations. It should be remarked that our main theorem does not work analogously for Ricci curvature since the n-th Hirzebruch surfaces $\mathbb{F}_n$ do not have positive Ricci curvature for $n \geq 2$ (see Proposition 4.7).

In Chapter 4, we first discuss an effective curvature pinching result for the holomorphic sectional curvature on projectivized rank 2 holomorphic vector bundles over $\mathbb{CP}^1$:

Theorem 1.3 ([ACH15]). *Let $\mathbb{F}_n$, $n \in \{1, 2, 3, \ldots\}$, be the n-th Hirzebruch surface. Then there exists a Hodge metric on $\mathbb{F}_n$ whose holomorphic sectional curvature is $\frac{1}{(1+2n)^2}$-pinched.*

We then generalize the case of the 0-th Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$ and prove the following result on products of Hermitian manifolds of positive holomorphic sectional curvature:

Theorem 1.4 ([ACH15]). *Let M and N be Hermitian manifolds whose positive holomorphic sectional curvatures are c_M- and c_N-pinched, respectively, and satisfy*

$$kc_M \leq K_M \leq k \text{ and } kc_N \leq K_N \leq k$$

for a constant $k > 0$. Then the holomorphic sectional curvature K of the product metric on $M \times N$ satisfies

$$k\frac{c_M c_N}{c_M + c_N} \leq K \leq k$$

and is $\frac{c_M c_N}{c_M + c_N}$-pinched.

This product result may seem surprising or unlikely due to the Hopf Conjecture in Riemannian geometry, which states the product of two real 2-spheres does not admit a Riemannian metric of positive sectional curvature. Because this conjecture is on [Riemannian] sectional curvature, the computation considers *all* planes inside the tangent space—not just holomorphic planes. Hence, there is no contradiction with our product result.

We remark that some of the results in this dissertation have appeared elsewhere. The work in Chapter 4 has been published in [ACH15] and the work in Chapter 3 is to appear in [AHZ15].

Chapter 2

Definitions and Preliminaries

2.1 Hermitian and Kähler Metrics

Definition 2.1. Let M be an n-dimensional complex manifold and let $p \in M$. A *Hermitian metric* on M is a positive-definite Hermitian inner product

$$g_p : T'_pM \otimes \overline{T'_pM} \to \mathbb{C}$$

which depends smoothly on $p \in M$.

Let U be a small open set in M such that $p \in U$. We say g_p "depends smoothly on $p \in M$" if $z = (z_1, ..., z_n)$ are local coordinates around p and $\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\}$ is the standard basis for the holomorphic tangent space T'_pM, then the functions

$$g_{i\bar{j}} : U \to \mathbb{C}, \quad p \mapsto g_p\left(\frac{\partial}{\partial z_i}(p), \frac{\partial}{\partial \bar{z}_j}(p)\right) \tag{2.1}$$

are smooth for all $i, j \in \{1, 2, ..., n\}$.

Let $\{dz_1, ..., dz_n\}$ be the dual basis of $\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\}$. Then locally, the Hermitian metric can be written as

$$g = \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \otimes d\bar{z}_j,$$

where the $g_{i\bar{j}}$ form an $n \times n$ positive definite Hermitian matrix $\left(g_{i\bar{j}}\right)$ of smooth functions defined in (2.1). The metric g can be decomposed into two parts:

1. The real part, denoted by $Re(g)$

2. The imaginary part, denoted by $Im(g)$.

The real part $Re(g)$ gives an ordinary inner product called the *induced Riemannian metric* of g. The imaginary part $Im(g)$ represents an alternating $\mathbb{R}$-differential 2-form. In particular, it is a $(1,1)$-form.

Remark 2.2. If we let $h := Re(g)$, then h is the Riemannian metric of the underlying smooth manifold $M_{\mathbb{R}}$ of M. Hence, every Hermitian manifold is also a Riemannian manifold. Unfortunately, not every Riemannian manifold is a Hermitian manifold. Given a Riemannian metric h on a complex manifold, then h “comes from” a Hermitian metric if it respects the complex structure; i.e., for all vector fields X, Y and the complex structure J, $h(JX, JY) = h(X, Y)$.

We can decompose our metric g as $g = Re(g) + \sqrt{-1}Im(g)$. Let $\omega := -\frac{1}{2}Im(g)$.

Definition 2.3. The $(1,1)$-form ω is called the *associated* $(1,1)$*-form of* g.

In coordinates, the associated $(1,1)$-form can be written locally as

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j. \tag{2.2}$$

Definition 2.4. The Hermitian metric g is called *Kähler* if ω is d-closed, where $d = \partial + \bar{\partial}$ is the exterior derivative.

The following proposition summarizes standard equivalences for a Hermitian metric being Kähler. In the proof of the proposition, we follow the exposition in [Bal06] and [Zhe00].

Proposition 2.5. *Let (M, g) be an n-dimensional Hermitian manifold. Then the following are equivalent:*

(i) g is Kähler (i.e., $d\omega = 0$).

(ii) For every point $p \in M$, there exists a neighborhood $U \ni p$ and a smooth, real-valued function $F : U \to \mathbb{R}$ such that $\omega = \sqrt{-1}\partial\bar{\partial}F$ on U. We call F the Kähler potential.

(iii) In any local coordinate system,

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, 1 \leq i, j, k \leq n,$$

or equivalently,

$$\frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} = \frac{\partial g_{i\bar{l}}}{\partial \bar{z}_j}, 1 \leq i, j, l \leq n.$$

(iv) For any point $p \in M$, there exist local holomorphic coordinates $(z_1, ..., z_n)$ in a neighborhood of p such that

$$g_{i\bar{j}}(p) = \delta_{ij} \quad \textit{and} \quad (dg_{i\bar{j}})(p) = 0.$$

Such coordinates are called normal coordinates.

Proof. We first prove that $\boxed{(i) \Longleftrightarrow (ii)}$.

"$\Longrightarrow$" Assume g is a Kähler metric; i.e., $d\omega = 0$. Because ω is closed, then we know that for a sufficiently small open set U, there exists a 1-form μ such that $d\mu = \omega$. Because μ is a 1-form, we can decompose μ into a $(1,0)$-form and $(0,1)$-form, precisely $\mu = \mu^{1,0} + \mu^{0,1}$. Then

$$\omega = d\mu = (\partial + \bar{\partial})(\mu^{1,0} + \mu^{0,1}) = \partial\mu^{1,0} + \partial\mu^{0,1} + \bar{\partial}\mu^{0,1} + \bar{\partial}\mu^{0,1} = \partial\mu^{0,1} + \bar{\partial}\mu^{1,0},$$

where $\partial\mu^{1,0} = \bar{\partial}\mu^{0,1} = 0$ since they are, respectively, $(2,0)$ and $(0,2)$-forms and we know ω is a $(1,1)$-form by definition. Because $\mu^{1,0}$ is ∂-closed and $\mu^{0,1}$ is $\bar{\partial}$-closed, we know that on our sufficiently small open set U, there exist smooth functions f_1 and f_2 such that $-\partial f_1 = \mu^{1,0}$ and $\bar{\partial} f_2 = \mu^{0,1}$. Because $\partial\bar{\partial} = -\bar{\partial}\partial$, we get

$$\sqrt{-1}\partial\bar{\partial}(f_2 + f_1) = \partial(\bar{\partial} f_2) + \bar{\partial}(-\partial f_1) = \partial\mu^{0,1} + \bar{\partial}\mu^{1,0} = d\mu = \omega.$$

Let $F := f_2 + f_1$. Because ω is a real $(1,1)$-form, we can assume F is real-valued.

"$\Longleftarrow$" Assume there exists a Kähler potential F. Then

$$d\omega = \sqrt{-1}d(\partial\bar{\partial}F) = \sqrt{-1}(\partial + \bar{\partial})(\partial\bar{\partial}F) = \sqrt{-1}(\partial\partial\bar{\partial}F - \partial\bar{\partial}\bar{\partial}F) = 0$$

since $\partial^2 = \bar{\partial}^2 = 0$.

To show $\boxed{(i) \Longleftrightarrow (iii)}$, we directly compute the derivatives and see that (iii) is just the local coordinate version of (i).

Lastly, we show that $\boxed{(i) \Longleftrightarrow (iv)}$. "$\Longrightarrow$" Assume (i) holds. After a possible constant linear change if necessary, we have that $g_{i\bar{j}}(p) = \delta_{ij}$, for all $i, j \in \{1, ..., n\}$. Define the constant matrix A^j by $A^j_{ik} = \frac{\partial g_{i\bar{j}}}{\partial z_k}(p)$, which is a symmetric matrix due to condition (iii). Define new holomorphic coordinates $(\tilde{z}_1, ..., \tilde{z}_n)$ by

$$\tilde{z}_j = z_j + \frac{1}{2} \sum_{i,k=1} A^j_{ik} z_i z_k. \tag{2.3}$$

Under these new coordinates, the metric can be represented by the matrix $\tilde{g} = B^{-1} g (B^{-1})^*$, where * denotes the conjugate transpose of B^{-1}, and the entries of B are

$$B_{ij} = \delta_{ij} + \sum_{k=1}^{n} A^j_{ik} z_k.$$

Direct computation shows that $(d\tilde{g})(p) = 0$.

"$\Longleftarrow$" Now assume there exist local holomorphic normal coordinates around each point $p \in M$. This means that there exist coordinates $(z_1, ..., z_n)$ such that $(dg)(p) = 0$. Because all derivatives at p are 0, we clearly must have $d\omega = 0$. $\square$

We observe that the equivalence of (i) and (iv) implies that a metric g is Kähler if and only if it can have second order approximation to the Euclidean metric at every point; i.e., g can be written as

$$g = \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \otimes d\bar{z}_j = \sum_{i,j=1}^{n} (\delta_{ij} + O(2)) dz_i \otimes d\bar{z}_j.$$

Additionally, any submanifold N of a Kähler manifold (M, g) is also Kähler since $d(\omega|_N) = (d\omega)|_N = 0$.

Example 2.6. Let $M = \mathbb{CP}^n$. Let $[w] = [w_0, ..., w_n]$ be homogeneous coordinates on M and let $|w|^2 = \sum_{i=0}^n |w_i|^2$. Note that for $\lambda \in \mathbb{C}^*$, $\log(|\lambda w|^2) = \log|\lambda|^2 + \log|w|^2$. Consider the form

$$\omega = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log|w|^2, \tag{2.4}$$

which is a well-defined, closed global $(1,1)$-form on $\mathbb{CP}^n$. To see that ω is positive-definite, take the standard coordinate charts on $\mathbb{CP}^n$, $\{U_i\}_{i=0}^n$, where

$$U_i = \{[w_0, ..., w_n] \mid w_i \neq 0\}.$$

In U_0, let $z = (z_1, ..., z_n)$ be a local affine coordinates where $z_i = \frac{w_i}{w_0}$, for $i = 1, ..., n$. Then $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. In U_0, we have that ω is given by

$$\frac{2}{\sqrt{-1}}\omega = \partial\bar{\partial}\log(1+|z|^2) = \sum_{i,j=1}^{n} \frac{(1+|z|^2)\delta_{ij} - \bar{z}_i z_j}{(1+|z|^2)^2} dz_i \wedge d\bar{z}_j. \tag{2.5}$$

Let $g = (g_{i\bar{j}})$, where $g_{i\bar{j}} = \frac{(1+|z|^2)\delta_{ij} - \bar{z}_i z_j}{(1+|z|^2)^2}$. Because the determinants of all the $k \times k$ minors of g are all positive for $1 \leq k \leq n$, we know by Sylvester's criterion that g is a positive-definite Hermitian matrix. Hence, ω is positive on U_0. Since U_0 can be replaced any U_i, $1 \leq i \leq n$, we have that ω is positive everywhere on $\mathbb{CP}^n$. In fact, ω is the associated $(1,1)$-form of a Kähler metric called the *Fubini-Study metric* of $\mathbb{CP}^n$.

Definition 2.7. A Hermitian metric is called a *Hodge metric* if it is Kähler and the cohomology class of its associated $(1,1)$-form is rational.

2.2 Connections on Complex Vector Bundles

We first discuss a few preliminary notions on a complex vector bundle $E \to M$ over a complex manifold M. In the next section, we will let $E = TM$ to define our curvatures. In this section, the notation $\mathcal{A}^p(E)$ will denote the sheaf of E-valued p-forms on M. Note that $\mathcal{A}^k(E) = \oplus_{p+q=k}\mathcal{A}^{p,q}(E)$, where $\mathcal{A}^{p,q}(E)$ is the sheaf of smooth E-valued (p,q)-forms on M.

Definition 2.8. Let $E \to M$ be a complex vector bundle. A *Hermitian metric* on E is a Hermitian inner product $\langle \cdot, \cdot \rangle$ on each fiber E_p of E which depends smoothly on $p \in M$.

Let $E \to M$ be a rank n vector bundle, where $n \in \mathbb{N}$. Let $\{e_1, ..., e_n\}$ be a local frame of E over an open set $U \subseteq M$; i.e., the set $\{e_1, ..., e_n\}$ forms a basis of sections for each fiber. By "depends smoothly", we mean that the functions $h_{i\bar{j}} = \langle e_i(p), e_j(p) \rangle$ are smooth.

Definition 2.9. Let M be a complex manifold and let $E \to M$ be a complex vector bundle of rank $n \in \mathbb{N}$. A *connection* on E is a linear map $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ which satisfies the Leibniz' rule: For all $f \in C^\infty(M)$ and for all $\xi \in \mathcal{A}^0(E)$,

$$D(f\xi) = df \otimes \xi + f \cdot D(\xi).$$

Using the local frame $\{e_1, ..., e_n\}$, a connection D can be locally written as $De_\alpha = \sum_{\beta=1}^n \theta_{\alpha\beta} e_\beta$, where the $\theta_{\alpha\beta}$ are 1-forms.

Definition 2.10. The matrix of 1-forms, $\theta = (\theta_{\alpha\beta})$, is called the *connection matrix* of D with respect to the local frame $\{e_1, ..., e_n\}$.

We will focus on a special kind of connection which satisfies two compatibility criteria.

Definition 2.11. Let D be a connection over a complex vector bundle E. Then D is *compatible with the complex structure* if the composition of maps

$$pr_2 \circ D : \mathcal{A}^0(E) \overset{D}{\to} \mathcal{A}^1(E) \overset{pr_2}{\to} \mathcal{A}^{0,1}(E)$$

is just equal to $\bar{\partial}$.

Definition 2.12. Let D be a connection over a complex vector bundle E and let $\langle \cdot, \cdot \rangle$ be a Hermitian metric on E. We say D is *compatible with the metric structure* if for all $\xi, \eta \in \mathcal{A}^0(E)$, $d\langle \xi, \mu \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle$.

Proposition 2.13. *Let E be a holomorphic vector bundle with Hermitian metric $\langle \cdot, \cdot \rangle$ on E. Then there exists uniquely a connection D that is compatible with both the complex structure and the metric structure.*

Definition 2.14. The unique connection of E compatible with both the complex structure and the metric is called the *canonical metric connection*, or the *Hermitian connection*.

Furthermore, we can define the following connection for $p \geq 1$:

$$D_p : \mathcal{A}^p(E) \to \mathcal{A}^{p+1}(E)$$

via the Leibniz rule: For all $\xi \in \mathcal{A}^0(E)$ and for all $\psi \in \mathcal{A}^p(M)$,

$$D_p(\psi \cdot \xi) = d\psi \otimes \xi + (-1)^p \psi \wedge D\xi.$$

In particular, we have that the map $D^2 = D \circ D : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$ is linear over $\mathcal{A}^0(E)$. In other words, $D^2(f\sigma) = fD^2(\sigma)$, for any $f \in C^\infty(M)$ and $\sigma \in \mathcal{A}^0(E)$. Hence, for any local frame $\{e_1, ..., e_n\}$ over an open set $U \subseteq M$, D^2 can locally be written as $D^2 e_\alpha = \sum_{\beta=1}^n \Theta_{\alpha\beta} e_\beta$, where the $\Theta_{\alpha\beta}$ are 2-forms.

Definition 2.15. The matrix of 2-forms, $\Theta = (\Theta_{\alpha\beta})$, is called the *curvature matrix* of D with respect to the local frame $\{e_1, ..., e_n\}$.

Remark 2.16. The curvature matrix Θ can be decomposed as $\Theta = \oplus_{p+q=2}\Theta^{p,q}$ where $\Theta^{p,q}$ is a matrix of (p,q)-forms. If D is compatible with the complex structure, then $\Theta^{0,2} = 0$. This is because $D^{0,1} = \bar{\partial}$, and $\sum_{\beta=1}^{n} \Theta^{0,2}_{\alpha\beta} e_\beta = D^{0,2} e_\alpha = (D^{0,1})^2 e_\alpha$. Additionally, if D is also compatible with the metric, we can choose a unitary frame of E such that both θ and Θ are skew-Hermitian—that is, $\theta^* = -\theta$ and $\Theta^* = -\Theta$, where "*" denotes the conjugate transpose. With Θ skew-Hermitian and $\Theta^{0,2} = 0$, we have that $\Theta^{2,0} = -(\Theta^{2,0})^* = -(\Theta^{0,2})^T = 0$. Hence, Θ consists of only $(1,1)$-forms.

2.3 The Curvatures of a Hermitian Metric

Let E be a rank $n \in \mathbb{N}$ holomorphic vector bundle and let u and v be sections of E. Let $\{e_1, ..., e_n\}$ be a local frame for E and assume that Θ is the curvature matrix of the canonical metric connection D. Write $u = \sum_{i=1}^{n} u_i e_i$ and $v = \sum_{i=1}^{n} v_i e_i$. Define the following $(1,1)$-form as follows:

$$\Theta_{u\bar{v}} = \sum_{i,j,k=1}^{n} \Theta_{ik} g_{k\bar{j}} u_i \bar{v}_j,$$

where $g_{k\bar{j}}$ is a Hermitian metric on E.

Definition 2.17. Let u and v be sections of E and let X and Y be tangent vectors on M. Define the 4-tensor $R_{X\bar{Y}u\bar{v}}$ by $R_{X\bar{Y}u\bar{v}} := \Theta_{u\bar{v}}(X, \bar{Y})$.

We observe that because Θ is skew-Hermitian and $\Theta_{v\bar{u}} = -\overline{\Theta_{u\bar{v}}}$, we have

$$\overline{R_{X\bar{Y}u\bar{v}}} = R_{Y\bar{X}v\bar{u}}.$$

2.3.1 The Components of the Curvature Tensor

In this section, we discuss the special case of a vector bundle E which is key for defining our curvatures. Let M be an n-dimensional Hermitian manifold with Hermitian metric g and canonical metric connection D. Let $E = TM$. Then, for any $(1,0)$-tangent vectors X, Y, Z, W, write

$$R(X, \bar{Y}, Z, \bar{W}) := R_{X\bar{Y}Z\bar{W}} = \Theta_{Z\bar{W}}(X\bar{Y}).$$

Let $\{e_1, ..., e_n\}$ be a local frame for TM, in which case we write $R_{i\bar{j}k\bar{l}} := R_{e_i\bar{e}_je_k\bar{e}_l}$.

Definition 2.18. The $R_{i\bar{j}k\bar{l}}$ are called the *components of the curvature tensor* associated with the metric connection.

If we consider holomorphic coordinates $(z_1, ..., z_n)$ and let $\{e_i\}_{i=1}^n = \{\frac{\partial}{\partial z_i}\}_{i=1}^n$, we can write the components as

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{n} g^{q\bar{p}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{j}}}{\partial \bar{z}_l}, \tag{2.6}$$

where $g^{q\bar{p}}$ is to be interpreted as $(g^{-1})_{pq}$. Moreover, when g is Kähler, the $R_{i\bar{j}k\bar{l}}$ satisfy the following symmetry condition:

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{i\bar{l}k\bar{j}}. \tag{2.7}$$

Remark 2.19. We have an equivalent definition of the components of the curvature tensor for an arbitrary tangent vector X. Namely, if $X = \sum_{i=1}^n X_i \frac{\partial}{\partial z_i}$ is a $(1,0)$-

tangent vector on M, then

$$R_{X\bar{X}X\bar{X}} = -g(X, \bar{X}, X, \bar{X}) + \sum_{a,b=1}^{n} g^{b\bar{a}} g(X, \bar{a}, X) g(b, \bar{X}, \bar{X}), \tag{2.8}$$

where $g(X, \bar{X}, X, \bar{X})$ is to be considered as the 4-tensor defined as follows: Let

$$g\left(\frac{\partial}{\partial z_i}, \frac{\bar{\partial}}{\partial z_j}, \frac{\partial}{\partial z_k}, \frac{\bar{\partial}}{\partial z_l}\right) := \sum_{i,j,k,l=1}^{n} \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \bar{\partial} \bar{z}_l}. \tag{2.9}$$

By multi-linearity of g, we know

$$g(X, \bar{X}, X, \bar{X}) = \sum_{i,j,k,l=1}^{n} X_i \bar{X}_j X_k \bar{X}_l g\left(\frac{\partial}{\partial z_i}, \frac{\bar{\partial}}{\partial z_j}, \frac{\partial}{\partial z_k}, \frac{\bar{\partial}}{\partial z_l}\right).$$

Hence,

$$g(X, \bar{X}, X, \bar{X}) := \sum_{i,j,k,l=1}^{n} X_i \bar{X}_j X_k \bar{X}_l \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \bar{\partial} \bar{z}_l}. \tag{2.10}$$

Additionally, $g(X, \bar{a}, X)$ is the 3-tensor defined as follows: Let

$$g\left(\frac{\partial}{\partial z_i}, \frac{\bar{\partial}}{\partial z_a}, \frac{\partial}{\partial z_j}\right) := \sum_{i,j=1}^{n} \frac{\partial g_{i\bar{a}}}{\partial z_j}. \tag{2.11}$$

By multi-linearity, we know

$$g(X, \bar{a}, X) = \sum_{i,j=1}^{n} X_i X_j g\left(\frac{\partial}{\partial z_i}, \frac{\bar{\partial}}{\partial z_a}, \frac{\partial}{\partial z_j}\right).$$

Thus,

$$g(X, \bar{a}, X) := \sum_{i,j=1}^{n} X_i X_j \frac{\partial g_{i\bar{a}}}{\partial z_j}. \tag{2.12}$$

The term $g(b, \bar{X}, \bar{X})$ is defined similarly. We will use this equivalent definition in

Chapter 3 since it lends itself better to approximating the curvature from below.

2.3.2 Definitions of Curvatures on a Hermitian Manifold

Using the components of the curvature tensor, we can make the following precise definitions:

Definition 2.20. If $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial z_i}$ is a nonzero $(1,0)$-tangent vector at $p \in M$, then the *holomorphic sectional curvature* in the direction of X, denoted by $K(X)$, is given by

$$K(X) = \left(2 \sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}}(p) X_i \bar{X}_j X_k \bar{X}_l \right) / \left(\sum_{i,j,k,l=1}^{n} g_{i\bar{j}} g_{k\bar{l}} X_i \bar{X}_j X_k \bar{X}_l \right). \tag{2.13}$$

Note that the holomorphic sectional curvature of X is invariant under multiplication of X with a real nonzero scalar. As a result, it suffices to use unit vectors for which the value of the denominator is 1.

Remark 2.21. By (2.8) we have an equivalent (and concise) definition of holomorphic sectional curvature: If $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial z_i}$ is a $(1,0)$-tangent vector on M, the *holomorphic sectional curvature* in the direction of X is

$$K(X) = \frac{R_{X\bar{X}X\bar{X}}}{|X|^4}, \tag{2.14}$$

where $|\cdot|^4$ is with respect to the metric g. We will use this definition in Chapter 3.

Example 2.22. With the Fubini-Study metric discussed in Example 2.6, $\mathbb{CP}^n$ has constant holomorphic sectional curvature equal to 4.

When we consider an orthonormal basis $\{u_1, ..., u_n\}$ of TM, we can take the trace of the components of the curvature tensor and obtain the following curvatures.

Definition 2.23. The *Ricci curvature* in the direction of X is

$$Ric(X) = \sum_{i,j=1}^{n} r_{i\bar{j}} X_i \bar{X}_j, \tag{2.15}$$

where $r_{i\bar{j}} = \sum_{k,l=1}^{n} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$ and $g^{k\bar{l}} = (g^{-1})_{l\bar{k}}$.

Remark 2.24. One can compute $r_{i\bar{j}}$ without the use of the curvature tensors. Given a Hermitian metric $g = (g_{k\bar{l}})$, we have

$$r_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\left(\det g_{k\bar{l}}\right)). \tag{2.16}$$

There are actually several ways to define the Ricci curvature of g. Using the equation in (2.16), we have what is called *first Ricci curvature of* g. For definitions of the other two Ricci curvatures, we refer the reader to [Zhe00, Section 7.6]. If M is a Kähler manifold, all of the different Ricci curvatures coincide.

Using the $r_{i\bar{j}}$ in (2.16), we can define the following real, closed, and globally defined $(1,1)$-form on M:

Definition 2.25. The *Ricci curvature form* of a Hermitian metric g, denoted by $\mathrm{Ric}(g)$ is defined as

$$\mathrm{Ric}(g) := \sqrt{-1} \sum_{i,j=1}^{n} r_{i\bar{j}} dz_i \wedge d\bar{z}_j. \tag{2.17}$$

Under the same orthonormal basis, taking the trace of the $r_{i\bar{j}}$ gives us:

Definition 2.26. The *scalar curvature* τ is defined to be

$$\tau = \sum_{i,j=1}^{n} g^{i\bar{j}} r_{i\bar{j}} = \sum_{i,j,k,l=1}^{n} g^{i\bar{j}} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}.$$

The Ricci form and scalar curvature are related by the following formula:

Proposition 2.27. *Let (M, g) be an n-dimensional Kähler manifold with associated $(1,1)$-form ω. Then*

$$\mathrm{Ric}(g) \wedge \omega^{n-1} = \frac{2}{n} \tau \omega^n.$$

One can easily see that $\mathrm{Ric}(g) > 0$ implies that $\tau > 0$. Lastly, we can consider the scalar curvature on the whole manifold by defining:

Definition 2.28. The *total scalar curvature* T is defined to be

$$T = \int_M \tau \omega^n,$$

where ω is the associated $(1,1)$-form of the Hermitian metric g and $n = \dim M$.

In the Kähler case, a result of Berger implies that the holomorphic sectional curvature and scalar curvature always have the same sign.

Proposition 2.29 ([Ber66])**.** *Let (M, g) be a Kähler manifold. Let K be the holomorphic sectional curvature of M and let τ be the scalar curvature of M.*

(i) If $K > 0$, then $\tau > 0$

(ii) If $K \geq 0$, then $\tau \geq 0$

(iii) If $K < 0$, then $\tau < 0$

(iv) If $K \leq 0$, then $\tau \leq 0$.

Furthermore, in the negative case, the relationship between holomorphic sectional curvature and total scalar curvature is discussed in the following theorem.

Theorem 2.30 ([HW12]). *Let M be a projective manifold with a Kähler metric of negative holomorphic sectional curvature. Then the total scalar curvature of any Kähler metric on M is negative.*

2.3.3 Curvature Pinching

Let M be a compact Hermitian manifold with holomorphic sectional curvature $K(X)$.

Definition 2.31. Let $c \in (0, 1]$. We say that the holomorphic sectional curvature is *c-pinched* if

$$\frac{\min_X K(X)}{\max_X K(X)} = c \ \ (\leq 1),$$

where the maximum and minimum are taken over all (unit) tangent vectors across M.

The pinching constant of a compact (Hermitian, Kähler, Riemannian, etc.) manifold can help determine some global properties of the manifold. For instance it was shown in [SS85] that if the holomorphic sectional curvature of a 6-dimensional connected complete non-Kähler, nearly Kähler manifold M is c-pinched, where $c > \frac{2}{5}$, then M is isometric to the 6-sphere of constant curvature $\frac{\tau}{30}$, where τ is the scalar curvature of M. Additionally, results on compact Riemannian manifolds whose sectional curvature is $\frac{1}{4}$-pinched are discussed in [BS08] and [BS09].

2.4 Curvature and the (Anti-)Canonical Bundle

There is a significant interplay between the curvature of a complex manifold and the various algebraic-geometric notions of positivity of its (anti-)canonical bundle. We first make the following precise definitions which will be utilized in Chapter 4.

Definition 2.32. Let M be a compact complex manifold and let $L \to M$ be a holomorphic line bundle on M. Then L is *positive* or *ample* if there exists a Hermitian metric h on L such that its curvature form $\frac{1}{2\pi i}\partial\bar{\partial}\log h$ is a positive-definite $(1,1)$-form.

Definition 2.33. Let M be a compact complex manifold. Then M is *projective* if it can be embedded into a complex projective space $\mathbb{CP}^n$.

Using the Kodaira Embedding Theorem, a manifold is projective if and only if it admits a positive line bundle $L \to M$. Also, by Chow's Theorem, a manifold is projective if and only if it can be defined as the zero locus in projective space of a finite number of homogeneous polynomials (i.e., it is projective algebraic).

An important special case is when $L = K_M$, where $K_M := \bigwedge^{\dim M} TM^*$ is the highest exterior power of the cotangent bundle of M.

Definition 2.34. We call K_M the *canonical bundle* of M and call its dual $-K_M$ the *anti-canonical bundle* of M.

Definition 2.35. If $-K_M$ is ample, then M is called a *Fano* manifold.

Definition 2.36. The *first Chern class of* M is defined to be $c_1^{\mathbb{R}}(M) := c_1^{\mathbb{R}}(-K_M)$.

Using the usual abuse of notation, we will drop the $^{\mathbb{R}}$. If K_M is ample, then $c_1(K_M) > 0$, and if $-K_M$ is ample, then $c_1(-K_M) > 0$. By the work of Chern, we know that the cohomology class of $\frac{1}{2\pi}\mathrm{Ric}$ is equal to $c_1(M)$. This equality, together with Yau's solution of the Calabi Conjecture found in [Yau78], yields:

Theorem 2.37. *There exists a Kähler metric with Ric* < 0 *if and only if* $c_1(K_M) > 0$. *There exists a Kähler metric with Ric* > 0 *if and only if* $c_1(-K_M) > 0$.

From the definition of holomorphic sectional curvature, it is clear that the curvature tensor determines the holomorphic sectional curvature. Conversely, from [KN69, Proposition 7.1], we know that the holomorphic sectional curvature determines the components of the curvature tensor. Be that as it may, the positivity or negativity properties of the holomorphic sectional curvature do not necessarily transfer to the Ricci curvature. In the positive case, the relationship between holomorphic sectional curvature and Ricci curvature is a bit more subtle and mysterious. In the negative case, it was proven in [HLW10] that if $\dim M \leq 3$ and if there exists a Kähler metric on M with negative holomorphic sectional curvature, then there exists a Kähler metric on M with negative Ricci curvature. More recently, it was proven by Wu and Yau that this statement holds true for a projective manifold of any dimension, namely:

Theorem 2.38 ([WY15])**.** *If a projective manifold M admits a Kähler metric whose holomorphic sectional curvature is negative everywhere, then the canonical bundle K_M is ample.*

Moreover, this result was extended by Tosatti and Yang in [TY15] from the projective case to the Kähler case:

Theorem 2.39 ([TY15])**.** *Let M be a compact Kähler manifold with negative holomorphic sectional curvature. Then K_M is ample.*

By Theorem 2.37, Theorem 2.38, and Theorem 2.39, we see that M having negative holomorphic sectional curvature implies that M also has negative Ricci curvature. In the positive case, it does not hold true that positive holomorphic sectional curvature implies positive Ricci curvature. For instance, the n-th Hirzebruch surface

$\mathbb{F}_n$ admits a Kähler metric of positive holomorphic sectional curvature, but does not admit a metric of positive Ricci curvature for $n \geq 2$. In Section 4.1, we briefly discuss the Ricci curvature of Hirzebruch surfaces using the notion of intersection numbers, which is defined as follows:

Definition 2.40. Let M be a compact complex manifold and let $D_1, ..., D_k$ be divisors on M. Then the *intersection number* of $D_1, ..., D_k$ and a k-dimensional subvariety V is

$$D_1. \cdots .D_k.V = \int_V c_1(D_1) \wedge \cdots \wedge c_1(D_k) \in \mathbb{Z}.$$

Using intersection numbers, we can state a numerical criterion for ampleness.

Theorem 2.41 ([Kle66], [Moi61], [Moi62], [Nak60], [Nak63])**.** *Let M be a projective manifold and let L be a line bundle on M. Let D be a divisor on M such that $L = [D]$. Then L is ample if and only if for all positive-dimensional irreducible subvarieties $V \subseteq M$,*

$$D^{\dim V}.V = \int_V c_1(D)^{\dim V} > 0.$$

Note that letting $V = M$ is allowed in Theorem 2.41. This theorem is called the Nakai-Moishezon-Kleiman criterion, but we will refer to this theorem as Nakai's criterion for the sake of brevity. A complete proof of this result can be found in [Laz04]. In particular, if M is a smooth projective surface, we have that L is ample if and only if its self-intersection number $L.L$ is positive, and for any irreducible curve C on M, $L.C > 0$.

Chapter 3

Metrics of Positive Curvature on Projectivized Vector Bundles

In this chapter, we prove a generalization of Hitchin's result on the positive holomorphic sectional curvature on Hirzebruch surfaces found in [Hit75]. We also provide a partial answer to the open question posed by Yau in Riemannian geometry in [SY10, Problem 6] by considering Yau's question in a complex setting and by considering projectivized vector bundles. This allows us to naturally replace "nonnegative sectional curvature" in Yau's question with "positive holomorphic sectional curvature". Under these circumstances, we arrive at our main theorem which is found in [AHZ15].

3.1 Proof of Theorem 1.2

We recall our main theorem on projectivized vector bundles over compact Kähler manifolds:

Theorem 1.2 ([AHZ15]). *Let M be an n-dimensional compact Kähler manifold.*

Let E be holomorphic vector bundle over M and let $\pi : P = \mathbb{P}(E) \to M$ be the projectivization of E. If M has positive holomorphic sectional curvature, then P admits a Kähler metric with positive holomorphic sectional curvature.

Proof. Let (M, g) be an n-dimensional compact Kähler manifold with positive holomorphic sectional curvature and let ω_g be the associated $(1,1)$-form of g. Let E be a rank $(r+1)$ vector bundle on M and let h be an arbitrary Hermitian metric on E. Let $(x, [v])$ be a moving point on P, where $x \in M$ and $[v] \in \mathbb{P}(E)$. The metrics g and h naturally induce a closed associated $(1,1)$-form on P:

$$\omega_G = \pi^*(\omega_g) + s\sqrt{-1}\partial\bar{\partial}\log h(v, \bar{v}), \tag{3.1}$$

which is the associated $(1,1)$-form of the metric $G := G_s$ on the total space. Write $h_{v\bar{v}} := h(v, \bar{v})$. For $s \in \mathbb{R}^+$ sufficiently small, ω_G is positive-definite everywhere. Thus, G is a Kähler metric on P.

We claim that for s sufficiently small (depending on g and h), the metric G has positive holomorphic sectional curvature. Fix an arbitrary point $p = (x_0, [w]) \in P$. Without loss of generality, let us assume that $|w| = 1$. Since M is assumed to be a Kähler manifold, we know by Proposition 2.5 that there exist holomorphic coordinates $z = (z_1, ..., z_n)$ centered at x_0 that are normal with respect to g; i.e., $x_0 = (0, ..., 0)$, and

$$g_{i\bar{j}}(0) = \delta_{ij}, \ \ (dg_{i\bar{j}})(0) = 0, \text{ for all } 1 \leq i, j \leq n.$$

Let Θ^h be the curvature form for the vector bundle (E, h). Using a constant unitary change of $(z_1, ..., z_n)$ if necessary, we may assume that the $(1,1)$-form $\Theta^h_{w\bar{w}} := \Theta^h(w, \bar{w})$ is diagonal. This means at x_0, Θ^h can be written as $\Theta^h_{w\bar{w}} = \sum_{i=1}^n \Theta_i dz_i \wedge d\bar{z}_i$.

Let $\{e_0, e_1, ..., e_r\}$ be a holomorphic local frame of E near x_0. We may assume that $e_0(0) = w$. We may also assume that

$$h_{\alpha\bar{\beta}}(0) = \delta_{\alpha\beta}, \ \ (dh_{\alpha\bar{\beta}})(0) = 0, \text{ for all } 1 \le \alpha, \beta \le r$$

and

$$\partial_i \partial_k h_{\alpha\bar{\beta}}(0) = \frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z_i \partial z_k}(0) = 0.$$

For $[v]$ in the moving point in P, we can use the holomorphic frame and write

$$v = e_0(z) + \sum_{\alpha=1}^{r} t_\alpha e_\alpha(z). \tag{3.2}$$

Thus, $(z, t) = (z_1, ..., z_n, t_1, ..., t_r)$ becomes local holomorphic coordinates in P centered at $p = (x_0, [w]) \in P$. Without loss of generality, we can shift our point p to the origin 0 and assume that our coordinates (z, t) are equal to $(0, 0)$.

We first want to compute G. After direct computation, we obtain

$$G = G_{i\bar{j}} dz_i \wedge d\bar{z}_j + G_{i\bar{\beta}} dz_i \wedge d\bar{t}_\beta + G_{\alpha\bar{j}} dt_\alpha \wedge d\bar{z}_j + G_{\alpha\bar{\beta}} dt_\alpha \wedge d\bar{t}_\beta,$$

where

$$\begin{aligned}
G_{i\bar{j}} &= g_{i\bar{j}} + \frac{-s}{(h_{v\bar{v}})^2} \frac{\partial h_{v\bar{v}}}{\partial z_i} \frac{\partial h_{v\bar{v}}}{\partial \bar{z}_j} + \frac{s}{h_{v\bar{v}}} \frac{\partial^2 h_{v\bar{v}}}{\partial z_i \partial \bar{z}_j} \\
G_{i\bar{\beta}} &= \frac{-s}{(h_{v\bar{v}})^2} \frac{\partial h_{v\bar{v}}}{\partial z_i} \frac{\partial h_{v\bar{v}}}{\partial \bar{t}_\beta} + \frac{s}{h_{v\bar{v}}} \frac{\partial^2 h_{v\bar{v}}}{\partial z_i \partial \bar{t}_\beta} \\
G_{\alpha\bar{j}} &= \frac{-s}{(h_{v\bar{v}})^2} \frac{\partial h_{v\bar{v}}}{\partial t_\alpha} \frac{\partial h_{v\bar{v}}}{\partial \bar{z}_j} + \frac{s}{h_{v\bar{v}}} \frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha \partial \bar{z}_j} \\
G_{\alpha\bar{\beta}} &= \frac{-s}{(h_{v\bar{v}})^2} \frac{\partial h_{v\bar{v}}}{\partial t_\alpha} \frac{\partial h_{v\bar{v}}}{\partial \bar{t}_\beta} + \frac{s}{h_{v\bar{v}}} \frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha \partial \bar{t}_\beta}.
\end{aligned}$$

Note that because h depends only on z_k and v is linear in t_β, we know that

$$\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial \bar{t}_\beta}(0) = \frac{\partial^2 h_{v\bar{v}}}{\partial t_\beta \partial \bar{z}_k}(0) = \frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial t_\beta}(0) = 0.$$

Thus, we have $G_{i\bar{j}}(0) = (\lambda - \Theta_i)\delta_{ij}$, $G_{i\bar{\beta}}(0) = G_{\alpha\bar{j}}(0) = 0$, and $G_{\alpha\bar{\beta}}(0) = \delta_{\alpha\beta}$. Hence, G is diagonal at p. In order to compute the components of the curvature tensor of G, we need to compute the derivatives up to second order. From (3.2), we can see that v is linear in t_α. Hence, any derivative of $h_{v\bar{v}}$ of second-order or higher with respect to t_α is identically equal to 0. The first derivatives of G are as follows:

$$\begin{aligned}\frac{\partial G_{i\bar{j}}}{\partial z_k} &= \frac{\partial g_{i\bar{j}}}{\partial z_k} + \frac{s}{h_{v\bar{v}}}\frac{\partial^3 h_{v\bar{v}}}{\partial z_k \partial z_i \partial \bar{z}_j} - \frac{s}{(h_{v\bar{v}})^2}\frac{\partial h_{v\bar{v}}}{\partial z_k}\frac{\partial^2 h_{v\bar{v}}}{\partial z_i \partial \bar{z}_j} + \frac{2s}{(h_{v\bar{v}})^3}\frac{\partial h_{v\bar{v}}}{\partial z_k}\frac{\partial h_{v\bar{v}}}{\partial z_i}\frac{\partial h_{v\bar{v}}}{\partial \bar{z}_j}\\ &- \frac{s}{(h_{v\bar{v}})^2}\left(\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial z_i}\frac{\partial h_{v\bar{v}}}{\partial \bar{z}_j} + \frac{\partial h_{v\bar{v}}}{\partial z_i}\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial \bar{z}_j}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial G_{i\bar{\beta}}}{\partial z_k} &= \frac{-s}{(h_{v\bar{v}})^2}\frac{\partial h_{v\bar{v}}}{\partial z_k}\frac{\partial^2 h_{v\bar{v}}}{\partial z_i \partial \bar{t}_\beta} + \frac{s}{h_{v\bar{v}}}\frac{\partial^3 h_{v\bar{v}}}{\partial z_k \partial z_i \partial \bar{t}_\beta} + \frac{2s}{(h_{v\bar{v}})^3}\frac{\partial h_{v\bar{v}}}{\partial z_k}\frac{\partial h_{v\bar{v}}}{\partial z_i}\frac{\partial h_{v\bar{v}}}{\partial \bar{t}_\beta}\\ &- \frac{s}{(h_{v\bar{v}})^2}\left(\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial z_i}\frac{\partial h_{v\bar{v}}}{\partial \bar{t}_\beta} + \frac{\partial h_{v\bar{v}}}{\partial z_i}\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial \bar{t}_\beta}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial G_{\alpha\bar{j}}}{\partial z_k} &= \frac{-s}{(h_{v\bar{v}})^2}\frac{\partial h_{v\bar{v}}}{\partial z_k}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha \partial \bar{z}_j} + \frac{s}{h_{v\bar{v}}}\frac{\partial^3 h_{v\bar{v}}}{\partial z_k \partial t_\alpha \partial \bar{z}_j} + \frac{2s}{(h_{v\bar{v}})^3}\frac{\partial h_{v\bar{v}}}{\partial z_k}\frac{\partial h_{v\bar{v}}}{\partial t_\alpha}\frac{\partial h_{v\bar{v}}}{\partial \bar{z}_j}\\ &- \frac{s}{(h_{v\bar{v}})^2}\left(\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial t_\alpha}\frac{\partial h_{v\bar{v}}}{\partial \bar{z}_j} + \frac{\partial h_{v\bar{v}}}{\partial t_\alpha}\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial \bar{z}_j}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial G_{\alpha\bar{\beta}}}{\partial z_k} &= \frac{-s}{(h_{v\bar{v}})^2}\frac{\partial h_{v\bar{v}}}{\partial z_k}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha \partial \bar{t}_\beta} + \frac{s}{h_{v\bar{v}}}\frac{\partial^3 h_{v\bar{v}}}{\partial z_k \partial t_\alpha \partial \bar{t}_\beta} + \frac{2s}{(h_{v\bar{v}})^3}\frac{\partial h_{v\bar{v}}}{\partial z_k}\frac{\partial h_{v\bar{v}}}{\partial t_\alpha}\frac{\partial h_{v\bar{v}}}{\partial \bar{t}_\beta}\\ &- \frac{s}{(h_{v\bar{v}})^2}\left(\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial t_\alpha}\frac{\partial h_{v\bar{v}}}{\partial \bar{t}_\beta} + \frac{\partial h_{v\bar{v}}}{\partial t_\alpha}\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial \bar{t}_\beta}\right)\end{aligned}$$

$$\frac{\partial G_{\alpha\bar{j}}}{\partial t_\gamma} = \frac{-s}{(h_{v\bar{v}})^3}\frac{\partial h_{v\bar{v}}}{\partial t_\gamma}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha \partial \bar{z}_j} + \frac{2s}{(h_{v\bar{v}})^3}\frac{\partial h_{v\bar{v}}}{\partial t_\gamma}\frac{\partial h_{v\bar{v}}}{\partial t_\alpha}\frac{\partial h_{v\bar{v}}}{\partial \bar{z}_j} - \frac{s}{(h_{v\bar{v}})^2}\frac{\partial h_{v\bar{v}}}{\partial t_\alpha}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma \partial \bar{z}_j}$$

$$\frac{\partial G_{\alpha\bar{\beta}}}{\partial t_\gamma} = \frac{-s}{(h_{v\bar{v}})^2}\frac{\partial h_{v\bar{v}}}{\partial t_\gamma}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha \partial \bar{t}_\beta} + \frac{2s}{(h_{v\bar{v}})^3}\frac{\partial h_{v\bar{v}}}{\partial t_\gamma}\frac{\partial h_{v\bar{v}}}{\partial t_\alpha}\frac{\partial h_{v\bar{v}}}{\partial \bar{t}_\beta} - \frac{s}{(h_{v\bar{v}})^2}\frac{\partial h_{v\bar{v}}}{\partial t_\alpha}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma \partial \bar{t}_\beta}.$$

Because G is a Kähler metric, we have that $\frac{\partial G_{i\bar{j}}}{\partial t_\alpha} = \frac{\partial G_{\alpha\bar{j}}}{\partial z_i}$ and $\frac{\partial G_{i\bar{\beta}}}{\partial z_\alpha} = \frac{\partial G_{\alpha\bar{\beta}}}{\partial z_i}$. Also,

$$h_{v\bar{v}}(0) = |w|^2 = 1,$$

and all first order derivatives of h and g are zero at the origin. By taking another derivative and evaluating at the origin, we get

$$\frac{\partial^2 G_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}(0) = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + s\left(-\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial \bar{z}_l}\frac{\partial h_{v\bar{v}}}{\partial z_i \partial \bar{z}_j} + \frac{\partial^4 h_{v\bar{v}}}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l} - \frac{\partial^2 h_{v\bar{v}}}{\partial z_i \partial \bar{z}_l}\frac{\partial^2 h_{v\bar{v}}}{\partial z_k \partial \bar{z}_j}\right)$$

$$\frac{\partial G_{i\bar{j}}}{\partial z_k \partial \bar{t}_\beta}(0) = s\frac{\partial^4 h_{v\bar{v}}}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{t}_\beta}$$

$$\frac{\partial^2 G_{i\bar{j}}}{\partial t_\alpha \partial \bar{t}_\beta}(0) = s\left(\frac{\partial^4 h_{v\bar{v}}}{\partial z_i \partial \bar{z}_j \partial t_\alpha \partial \bar{t}_\beta} - \frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha \partial \bar{t}_\beta}\frac{\partial h^2 h_{v\bar{v}}}{\partial z_i \partial \bar{z}_j}\right)$$

$$\frac{\partial^2 G_{\alpha\bar{j}}}{\partial t_\gamma \partial \bar{z}_l}(0) = 0$$

$$\frac{\partial^2 G_{\alpha\bar{\beta}}}{\partial t_\gamma \partial \bar{z}_j}(0) = 0$$

$$\frac{\partial^2 G_{\alpha\bar\beta}}{\partial t_\gamma \partial \bar t_\delta}(0) = s\left(-\frac{\partial^2 h_{v\bar v}}{\partial t_\gamma \partial \bar t_\delta}\frac{\partial^2 h_{v\bar v}}{\partial t_\alpha \partial \bar t_\beta} - \frac{\partial^2 h_{v\bar v}}{\partial t_\alpha \partial \bar t_\delta}\frac{\partial^2 h_{v\bar v}}{\partial t_\gamma \partial \bar t_\beta}\right).$$

Let $X = \sum_{i=1}^n X_i \frac{\partial}{\partial z_i}$, $U = \sum_{\alpha=1}^r U_\alpha \frac{\partial}{\partial t_\alpha}$, and $V := X + U \in T_pP$. Let R be the curvature tensor of G, R^g be the curvature tensor of g, and R^h be the curvature tensor of h. Note that because the matrix of G is diagonal at p, the numerator of the holomorphic sectional curvature in terms of (2.8) is equal to

$$\begin{aligned} R_{V\bar V V\bar V} &= -G(V,\bar V,V,\bar V) + \sum_{a=1}^{n+r}\frac{1}{G_{a\bar a}}G(V,\bar a,V)G(a,\bar V,V) \\ &= -G(V,\bar V,V,\bar V) + \sum_{a=1}^{n+r}\frac{1}{G_{a\bar a}}|G(V,\bar a,V)|^2. \end{aligned} \tag{3.3}$$

Because the second summand is always nonnegative, we know

$$R_{V\bar V V\bar V} \geq -G(V,\bar V,V,\bar V).$$

Using the multi-linearity of G, we have that

$$\begin{aligned} G(V,\bar V,V,\bar V) &= G(X+U,\bar X+\bar U,X+U,\bar X+\bar U) \\ &= G(X,\bar X,X,\bar X) + G(U,\bar X,X,\bar X) + G(X,\bar U,X,\bar X) + G(U,\bar U,X,\bar X) \\ &+ G(X,\bar X,U,\bar X) + G(U,\bar X,U,\bar X) + G(X,\bar U,U,\bar X) + G(U,\bar U,U,\bar X) \\ &+ G(X,\bar X,X,\bar U) + G(U,\bar X,X,\bar U) + G(X,\bar U,X,\bar U) + G(U,\bar U,X,\bar U) \\ &+ G(X,\bar X,U,\bar U) + G(U,\bar X,U,\bar U) + G(X,\bar U,U,\bar U) + G(U,\bar U,U,\bar U). \end{aligned}$$

Note that $G(U,\bar U,X,\bar X)$ and $G(X,\bar X,U,\bar U)$ are real since they are equal to their own conjugates. Additionally,

$$G(U,\bar{X},X,\bar{X}) = \overline{G(X,\bar{U},X,\bar{X})},\;\; G(X,\bar{X},U,\bar{X}) = \overline{G(X,\bar{X},X,\bar{U})},$$

$$G(U,\bar{U},X,\bar{U}) = \overline{G(U,\bar{U},U,\bar{X})},\;\; G(X,\bar{U},U,\bar{U}) = \overline{G(U,\bar{X},U,\bar{U})},$$

$$G(U,\bar{X},U,\bar{X}) = \overline{G(X,\bar{U},X,\bar{U})},\;\; G(U,\bar{X},X,\bar{U}) = \overline{G(X,\bar{U},U,\bar{X})}.$$

Using Proposition 2.5 (iii), we also have that

$$G(U,\bar{X},X,\bar{X}) = G(X,\bar{X},U,\bar{X}),\;\; G(U,\bar{U},\bar{X},\bar{U}) = G(X,\bar{U},U,\bar{U}),$$

$$G(U,\bar{X},X,\bar{U}) = G(X,\bar{X},U,\bar{U}) = G(U,\bar{U},X,\bar{X}) = G(X,\bar{U},U,\bar{X}),$$

which shows that $G(U,\bar{X},X,\bar{U})$ is also real since it is equal to its conjugate. Hence, we have

$$\begin{aligned} G(V,\bar{V},V,\bar{V}) &= G(X,\bar{X},X,\bar{X}) + 4G(X,\bar{X},U,\bar{U}) + G(U,\bar{U},U,\bar{U}) \\ &+ 2Re(G(X,\bar{U},X,\bar{U})) + 2(2Re(G(X,\bar{X},X,\bar{U}))) \\ &+ 2(2Re(G(U,\bar{U},U,\bar{X}))) \\ &= G(X,\bar{X},X,\bar{X}) + 4G(X,\bar{X},U,\bar{U}) + G(U,\bar{U},U,\bar{U}) \\ &+ 2Re\left[G(X,\bar{U},X,\bar{U}) + 2G(X,\bar{X},X,\bar{U}) + 2G(U,\bar{U},U,\bar{X})\right]. \end{aligned}$$

By (2.9) and (2.10), we obtain the following:

$$\begin{aligned} &R_{V\bar{V}V\bar{V}} \geq -G(V,\bar{V},V,\bar{V}) \\ &= -G(X,\bar{X},X,\bar{X}) - 4G(X,\bar{X},U,\bar{U}) - G(U,\bar{U},U,\bar{U}) \end{aligned}$$

$$
\begin{aligned}
&- 2Re\left[G(X,\bar{U},X,\bar{U}) + 2G(X,\bar{X},X,\bar{U}) + 2G(U,\bar{U},U,\bar{X})\right] \\
&- 2Re\left[G\left(X_j\frac{\partial}{\partial z_j},\bar{U}_\alpha\frac{\bar{\partial}}{\partial t_\alpha},X_l\frac{\partial}{\partial z_l},\bar{U}_\gamma\frac{\bar{\partial}}{\partial t_\gamma}\right) + 2G\left(X_i\frac{\partial}{\partial z_i},\bar{X}_j\frac{\bar{\partial}}{\partial z_j},X_k\frac{\partial}{\partial z_k},\bar{U}_\beta\frac{\bar{\partial}}{\partial t_\beta}\right)\right] \\
&- 2Re\left[2G\left(U_\alpha\frac{\partial}{\partial t_\alpha},\bar{U}_\beta\frac{\bar{\partial}}{\partial t_\beta},U_\gamma\frac{\partial}{\partial t_\gamma},\bar{X}_j\frac{\bar{\partial}}{\partial z_j}\right)\right] \\
&= \sum_{i,j,k,l=1}^{n}\sum_{\alpha,\beta,\gamma,\delta=1}^{r} -X_i\bar{X}_jX_k\bar{X}_lG\left(\frac{\partial}{\partial z_i},\frac{\bar{\partial}}{\partial z_j},\frac{\partial}{\partial z_k},\frac{\bar{\partial}}{\partial z_l}\right) \\
&- 4X_i\bar{X}_jU_\alpha\bar{U}_\beta G\left(\frac{\partial}{\partial z_i},\frac{\bar{\partial}}{\partial z_j},\frac{\partial}{\partial t_\alpha},\frac{\bar{\partial}}{\partial t_\beta}\right) - U_\alpha\bar{U}_\beta U_\gamma\bar{U}_\delta G\left(\frac{\partial}{\partial t_\alpha},\frac{\bar{\partial}}{\partial t_\beta},\frac{\partial}{\partial t_\gamma},\frac{\bar{\partial}}{\partial t_\delta}\right) \\
&- 2Re\left[X_j\bar{U}_\alpha X_l\bar{U}_\gamma G\left(\frac{\partial}{\partial z_j},\frac{\bar{\partial}}{\partial t_\alpha},\frac{\partial}{\partial z_l},\frac{\bar{\partial}}{\partial t_\gamma}\right) + 2X_i\bar{X}_jX_k\bar{U}_\beta G\left(\frac{\partial}{\partial z_i},\frac{\bar{\partial}}{\partial z_j},\frac{\partial}{\partial z_k},\frac{\bar{\partial}}{\partial t_\beta}\right)\right] \\
&- 2Re\left[2U_\alpha\bar{U}_\beta U_\gamma\bar{X}_jG\left(\frac{\partial}{\partial t_\alpha},\frac{\bar{\partial}}{\partial t_\beta},\frac{\partial}{\partial t_\gamma},\frac{\bar{\partial}}{\partial z_j}\right)\right] \\
&= \sum_{i,j,k,l=1}^{n}\sum_{\alpha,\beta,\gamma,\delta=1}^{r} -X_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2 G_{i\bar{j}}}{\partial z_k\partial\bar{z}_l} - 4X_i\bar{X}_jU_\alpha\bar{U}_\beta\frac{\partial^2 G_{i\bar{j}}}{\partial t_\alpha\partial\bar{t}_\beta} - U_\alpha\bar{U}_\beta U_\gamma\bar{U}_\delta\frac{\partial^2 G_{\alpha\bar{\beta}}}{\partial t_\gamma\partial\bar{t}_\delta} \\
&- 2Re\left[X_j\bar{U}_\alpha X_l\bar{U}_\gamma\frac{\partial^2 G_{j\bar{\alpha}}}{\partial z_l\partial\bar{t}_\gamma} + 2X_i\bar{X}_jX_k\bar{U}_\beta\frac{\partial^2 G_{i\bar{j}}}{\partial z_k\partial\bar{t}_\beta} + 2U_\alpha\bar{U}_\beta U_\gamma\bar{X}_j\frac{\partial^2 G_{\alpha\bar{\beta}}}{\partial t_\gamma\partial\bar{z}_j}\right].
\end{aligned}
$$

Note that $\frac{\partial^2 G_{j\bar{\alpha}}}{\partial z_l\partial\bar{t}_\gamma}$ and $\frac{\partial^2 G_{\alpha\bar{j}}}{\partial t_\gamma\partial\bar{z}_l}$ are conjugates. Because $\frac{\partial^2 G_{\alpha\bar{j}}}{\partial t_\gamma\partial\bar{z}_l}$ is 0 at p, we have that $\frac{\partial^2 G_{\alpha\bar{j}}}{\partial t_\gamma\partial\bar{z}_l}$ is also 0 at p. Also, we computed $\frac{\partial^2 G_{\alpha\bar{\beta}}}{\partial t_\gamma\partial\bar{z}_j}$ to be 0 at p. Hence, we have

$$
\begin{aligned}
R_{V\bar{V}V\bar{V}} &\geq -G(V,\bar{V},V,\bar{V}) \\
&= \sum_{i,j,k,l=1}^{n}\sum_{\alpha,\beta,\gamma,\delta=1}^{r} -X_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2 G_{i\bar{j}}}{\partial z_k\partial\bar{z}_l} - 4X_i\bar{X}_jU_\alpha\bar{U}_\beta\frac{\partial^2 G_{1\bar{j}}}{\partial t_\alpha\partial\bar{t}_\beta} \\
&- U_\alpha\bar{U}_\beta U_\gamma\bar{U}_\delta\frac{\partial^2 G_{\alpha\bar{\beta}}}{\partial t_\gamma\partial\bar{t}_\delta} - 4Re\left[X_i\bar{X}_jX_k\bar{U}_\beta\frac{\partial^2 G_{i\bar{j}}}{\partial z_k\partial\bar{t}_\beta}\right].
\end{aligned}
\tag{3.4}
$$

Substituting the derivatives of G into (3.4) yields

$$
R_{V\bar{V}V\bar{V}} \geq \sum_{i,j,k,l=1}^{n}\sum_{\alpha,\beta,\gamma,\delta=1}^{r} -X_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2 g_{i\bar{j}}}{\partial z_k\partial\bar{z}_l}
$$

$$
\begin{aligned}
&- X_i\bar{X}_jX_k\bar{X}_ls\left(-\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{z}_l}\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j}+\frac{\partial^4 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j\partial z_k\partial\bar{z}_l}-\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_l}\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{z}_j}\right)\\
&- 4X_i\bar{X}_jU_\alpha\bar{U}_\beta s\left(\frac{\partial^4 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j\partial t_\alpha\partial\bar{t}_\beta}-\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta}\frac{\partial h^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j}\right)\\
&- U_\alpha\bar{U}_\beta U_\gamma\bar{U}_\delta s\left(-\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\delta}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta}-\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\delta}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\beta}\right)\\
&- 4sRe\left[X_i\bar{X}_jX_k\bar{U}_\beta\frac{\partial^4 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j\partial z_k\partial\bar{t}_\beta}\right]\\
&= \sum_{i,j,k,l=1}^{n}\sum_{\alpha,\beta,\gamma,\delta=1}^{r} -X_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2 g_{i\bar{j}}}{\partial z_k\partial\bar{z}_l}+sX_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{z}_l}\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j}\\
&+ sX_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_l}\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{z}_j}-sX_i\bar{X}_jX_k\bar{X}_l\frac{\partial^4 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j\partial z_k\partial\bar{z}_l}\\
&- 4sX_i\bar{X}_jU_\alpha\bar{U}_\beta\frac{\partial^4 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j\partial t_\alpha\partial\bar{t}_\beta}+4sX_i\bar{X}_jU_\alpha\bar{U}_\beta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta}\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j}\\
&+ sU_\alpha\bar{U}_\beta U_\gamma\bar{U}_\delta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\delta}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta}+sU_\alpha\bar{U}_\beta U_\gamma\bar{U}_\delta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\delta}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\beta}\\
&- 4sRe\left[X_i\bar{X}_jX_k\bar{U}_\beta\frac{\partial^4 h_{v\bar{v}}}{\partial z_i\bar{\partial} z_j\partial z_k\partial\bar{t}_\beta}\right].
\end{aligned}
$$

Note that $v\bar{v} = |v|^2 = h_{v\bar{v}}(0) = 1$. Also, because the first derivatives of h and g are all 0 at p, and the second summands of R^g and R^h given by (2.8) are equal to zero.

By (2.10), we have the following equalities:

$$\sum_{i,j,k,l=1}^{n} -X_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2 g_{i\bar{j}}}{\partial z_k\partial\bar{z}_l} = -g(X,\bar{X},X,\bar{X}) = R^g_{X\bar{X}X\bar{X}}$$

$$\begin{aligned}
\sum_{i,j,k,l=1}^{n} -X_i\bar{X}_jX_k\bar{X}_l\frac{\partial^4 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j\partial z_k\partial\bar{z}_l} &= \sum_{i,j,k,l=1}^{n} -X_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\left(\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{z}_l}\right)\\
&= \sum_{i,j,k,l=1}^{n} X_i\bar{X}_jX_k\bar{X}_l\left(-\frac{\partial^2 h_{k\bar{l}}}{\partial z_i\partial\bar{z}_j}\right)\\
&= -h(X,\bar{X},X,\bar{X}) = R^h_{X\bar{X}X\bar{X}}
\end{aligned}$$

$$\begin{aligned}
\sum_{i,j,k,l=1}^{n} X_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j}\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{z}_l} &= \sum_{i,j,k,l=1}^{n} X_i\bar{X}_j\left(-\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j}\right)X_k\bar{X}_l\left(-\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{z}_l}\right)\\
&= (-h(v,\bar{v},X,\bar{X}))^2 = (R^h_{v\bar{v}X\bar{X}})^2
\end{aligned}$$

$$\begin{aligned}
\sum_{i,j,k,l=1}^{n} X_i\bar{X}_jX_k\bar{X}_l\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_l}\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{z}_j} &= \sum_{i,j,k,l=1}^{n} X_i\bar{X}_l\left(-\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_l}\right)X_k\bar{X}_j\left(-\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{z}_j}\right)\\
&= (-h(v,\bar{v},X,\bar{X}))^2 = (R^h_{v\bar{v}X\bar{X}})^2
\end{aligned}$$

$$\begin{aligned}
\sum_{\alpha,\beta,\gamma,\delta=1}^{r} U_\alpha\bar{U}_\beta U_\gamma\bar{U}_\delta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\delta}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta} &= \sum_{\alpha,\beta,\gamma,\delta=1}^{r}\left(U_\alpha\bar{U}_\beta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta}\right)\left(U_\gamma\bar{U}_\delta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\delta}\right)\\
&= \left(\sum_{\alpha,\beta=1}^{r} U_\alpha\bar{U}_\beta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta}\right)\left(\sum_{\gamma,\delta=1}^{r} U_\gamma\bar{U}_\delta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\delta}\right)\\
&= |U|^2|U|^2 = |U|^4
\end{aligned}$$

$$\begin{aligned}
\sum_{\alpha,\beta,\gamma,\delta=1}^{r} U_\alpha\bar{U}_\beta U_\gamma\bar{U}_\delta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\delta}\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\beta} &= \sum_{\alpha,\beta,\gamma,\delta=1}^{r}\left(U_\alpha\bar{U}_\delta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\delta}\right)\left(U_\gamma\bar{U}_\beta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\beta}\right)\\
&= \left(\sum_{\alpha,,\gamma,\delta=1}^{r} U_\alpha\bar{U}_\delta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\delta}\right)\left(\sum_{\alpha,\beta,\gamma,\delta=1}^{r} U_\gamma\bar{U}_\beta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\gamma\partial\bar{t}_\beta}\right)\\
&= |U|^2|U|^2 = |U|^4
\end{aligned}$$

$$\begin{aligned}
\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{r} -X_i\bar{X}_jU_\alpha\bar{U}_\beta\frac{\partial^4 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j\partial t_\alpha\partial\bar{t}_\beta} &= \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{r} -X_i\bar{X}_jU_\alpha\bar{U}_\beta\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\left(\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta}\right)\\
&= \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{r} X_i\bar{X}_jU_\alpha\bar{U}_\beta\left(-\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z_i\partial\bar{z}_j}\right)\\
&= -h(U,\bar{U},X,\bar{X}) = R^h_{U\bar{U}X\bar{X}}\\
\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{r} X_i\bar{X}_jU_\alpha\bar{U}_\beta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta}\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j} &= \left(\sum_{\alpha,\beta=1}^{r} U_\alpha\bar{U}_\beta\frac{\partial^2 h_{v\bar{v}}}{\partial t_\alpha\partial\bar{t}_\beta}\right)\left(\sum_{i,j=1}^{n} X_i\bar{X}_j\frac{\partial^2 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j}\right)\\
&= |U|^2h(v,\bar{v},X,\bar{X}) = -|U|^2R^h_{v\bar{v}X\bar{X}}.
\end{aligned}$$

For the $Re(\cdot)$ part, we have

$$\begin{aligned}
&\sum_{i,j,k=1}^{n}\sum_{\beta=1}^{r} Re\left(X_i\bar{X}_jX_k\bar{U}_\beta\frac{\partial^4 h_{v\bar{v}}}{\partial z_i\partial\bar{z}_j\partial z_k\partial\bar{t}_\beta}\right)\\
&\quad= \sum_{i,j,k=1}^{n}\sum_{\beta=1}^{r} Re\left(X_i\bar{X}_jX_k\bar{U}_\beta\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\left(\frac{\partial^2 h_{v\bar{v}}}{\partial z_k\partial\bar{t}_\beta}\right)\right)\\
&\quad= \sum_{i,j,k=1}^{n}\sum_{\beta=1}^{r} Re\left(X_i\bar{X}_jX_k\bar{U}_\beta\frac{\partial^2 h_{k\bar{\beta}}}{\partial z_i\partial\bar{z}_j}\right)\\
&\quad= Re\left(\sum_{i,j,k=1}^{n}\sum_{\beta=1}^{r} X_i\bar{X}_jX_k\bar{U}_\beta\frac{\partial^2 h_{k\bar{\beta}}}{\partial z_i\partial\bar{z}_j}\right)\\
&\quad= Re(h(X,\bar{U},X,\bar{X})) = Re(-R^h_{X\bar{U}X\bar{X}}).
\end{aligned}$$

After substituting the equalities, we have that $R_{V\bar{V}V\bar{V}}$ is bounded below as follows:

$$\begin{aligned}
R_{V\bar{V}V\bar{V}} &\geq R^g_{X\bar{X}X\bar{X}} + s\left(R^h_{X\bar{X}X\bar{X}} + 2(R^h_{v\bar{v}X\bar{X}})^2\right)\\
&+ s\left(2|U|^4 + 4R^h_{U\bar{U}X\bar{X}} - 4|U|^2R^h_{v\bar{v}X\bar{X}} + 4Re(R^h_{X\bar{U}X\bar{X}})\right).
\end{aligned} \tag{3.5}$$

Let $H_0 \in \mathbb{R}^+$ be the minimum holomorphic sectional curvature of g on M—that is,

$$\frac{R^g_{X\bar{X}X\bar{X}}}{|X|^4} \geq H_0.$$

Due to the compactness of M, we know that there exists $C \in \mathbb{R}^+$ such that $|R^h| \leq C$. Under the assumption that $h_{v\bar{v}}(0) = |v|^2 = 1$, we have

$$\begin{aligned} R_{V\bar{V}V\bar{V}} &\geq H_0|X|^4 - sC|X|^4 + 2sC^2|X|^4 + 2s|U|^4 + 4s(-C|U|^2|X|^2) \\ &\quad - 4sC|U|^2|X|^2 + 4sRe(-C|X|^3|U|) \\ &= H_0|X|^4 + s\left(-C|X|^4 + 2C^2|X|^4 + 2|U|^4 - 8C|U|^2|X|^2 - 4C|X|^3|U|\right). \end{aligned}$$

Thus, the holomorphic sectional curvature of P is bounded below by a homogeneous degree 4 polynomial in variables $|X|$ and $|U|$. Call this polynomial $f := f(|X|, |U|)$. We show that when at least one of X and U is nonzero, f is positive for s sufficiently small. Consider the following two cases:

(i) When $X = 0$ and $U \neq 0$: $f(0, |U|) = 2s|U|^4$, which is positive.

(ii) When $X \neq 0$: Since f is homogeneous, we know that $f(|X|, |U|) = |X|^4 f(1, \tilde{U})$, where $\tilde{U} := \frac{|U|}{|X|}$. Thus, it suffices to check if $f(1, \tilde{U})$ is positive. Note that

$$f(1, \tilde{U}) = H_0 + s\left(2\tilde{U}^4 - 8C\tilde{U}^2 - 4C\tilde{U} + C'\right), \tag{3.6}$$

where $C' = 2C^2 - C$. Since the leading term inside the parentheses of (3.6) guarantees a minimum, we know that for sufficiently small s, $f(1, \tilde{U})$ is positive.

Hence, $R_{V\bar{V}V\bar{V}} > 0$ when $V \neq 0$. □

3.2 On the Grassmannian Bundle $G_k(E)$

If we replace $\mathbb{P}(E)$ in Theorem 1.2 with the k-Grassmannian bundle $G_k(E)$ of all k-dimensional subspaces of the fibers of E, then $G_k(E)$ also has positive holomorphic sectional curvature. Let E be a rank r vector bundle over a compact Kähler manifold M, where $r \geq k$. Let $(x_0, [v]) \in G_k(E)$ be a moving point and let $\{e_1, e_1, ..., e_r\}$ be a holomorphic local frame of E near x_0. The analogue to equation (3.2) is

$$v = e_1(z) + ... + e_k(z) + \sum_{\alpha=1}^{r-k} t_\alpha e_\alpha(z).$$

Thus, $(z, t) = (z_1, ..., z_n, t_1, ..., t_{r-k})$ becomes local holomorphic coordinates in $G_k(E)$ centered at our fixed point $p = (x_0, [w]) \in G_k(E)$. By using the methods from the proof of Theorem 1.2, we would arrive at $G_k(E)$ also having positive holomorphic sectional curvature.

Chapter 4

Curvature Pinching for Projectivized Vector Bundles over $\mathbb{CP}^1$

With the result in Theorem 1.2, it is natural to attempt to determine the pinching constants of the holomorphic sectional curvature of a projectivized vector bundle $\mathbb{P}(E)$. We first briefly discuss the concepts of decomposable and indecomposable vector bundles.

Definition 4.1. Let M be a complex manifold. A vector bundle $E \to M$ is *indecomposable* if it is not the direct sum of two vector bundles of smaller rank. We say E is *decomposable* if it is not indecomposable.

A quintessential example of an indecomposable vector bundle is the tangent bundle of $\mathbb{CP}^n$, when $n \geq 2$. It was proven by Horrocks in [Hor64] that any vector bundle on $\mathbb{CP}^n, n \geq 3$, is decomposable if and only if its restriction to a hyperplane $H = \mathbb{CP}^{n-1} \subset$

$\mathbb{CP}^n$ is decomposable. If M is $\mathbb{CP}^1$, then any vector bundle on M is decomposable and is equal to the direct sum of line bundles. More precisely, we have the following theorem:

Theorem 4.2 ([Bir09], [Gro57])**.** *Let E be a rank k holomorphic vector bundle over $\mathbb{CP}^1$, where $k \geq 1$. Then E is isomorphic to a direct sum of line bundles, namely*

$$E = \oplus_{i=1}^{k} \mathcal{O}_{\mathbb{CP}^1}(n_i), \quad n_i \in \mathbb{Z},$$

where $\mathcal{O}_{\mathbb{CP}^1}(n_i)$ represents the line bundle of degree n_i over $\mathbb{CP}^1$.

As a result, our primary stepping stone consists of studying pinching constants of $\mathbb{P}(E)$ where the base manifold is $\mathbb{CP}^1$.

We note that if L a line bundle, then $\mathbb{P}(E \otimes L) \cong \mathbb{P}(E)$. Thus, if E is a vector bundle of rank $k \geq 1$ over $\mathbb{CP}^1$ and if we tensor E by the line bundle

$$\mathcal{O}_{\mathbb{CP}^1}(-\min\{n_i \mid i = 1, ..., k\}),$$

the structure of the projectivized vector bundle is not altered. After tensoring by $\mathcal{O}_{\mathbb{CP}^1}(-\min\{n_i \mid i = 1, ..., k\})$, the projectivization of E can be written as

$$\mathbb{P}(E) = \mathbb{P}\left(\mathcal{O}_{\mathbb{CP}^1} \oplus \left(\oplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{CP}^1}(m_i)\right)\right), \ m_i \in \mathbb{Z}^{\geq 0},$$

where $\mathcal{O}_{\mathbb{CP}^1}$ denotes the trivial line bundle of $\mathbb{CP}^1$. The (optimal) pinching constants of $\mathbb{P}(E)$ will then depend on the nonnegative integers $m_1, ..., m_{k-1}$, which we call the "twisting" parameters.

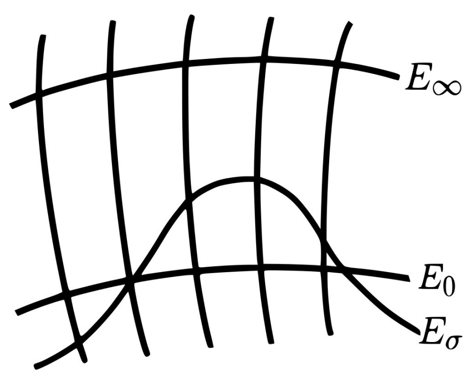
4.1 Curvature Pinching for Hirzebruch Surfaces

We consider the projectivization of a rank $k = 2$ vector bundle $\mathcal{O}_{\mathbb{CP}^1}(n) \oplus \mathcal{O}_{\mathbb{CP}^1}$, where $n \in \mathbb{Z}^{\geq 0}$.

Definition 4.3. The *n-th Hirzebruch Surface* is defined to be

$$\mathbb{F}_n := \mathbb{P}\left(\mathcal{O}_{\mathbb{CP}^1}(n) \oplus \mathcal{O}_{\mathbb{CP}^1}\right).$$

Let $(s_n, 1)$ be a section of $\mathcal{O}_{\mathbb{CP}^1}(n) \oplus \mathcal{O}_{\mathbb{CP}^1}$. After projectivizing the fibers, we have a section σ of $\mathbb{F}_n$. Let E_σ be the image of σ in $\mathbb{F}_n$. We then have the special curves on $\mathbb{F}_n$, E_0, E_σ, and E_∞, as shown below.



Regarding intersection numbers of these curves, we have

$$E_0.E_0 = n, \quad E_0.E_\infty = 0, \quad E_\infty.E_\infty = -n. \tag{4.1}$$

A more detailed description of this rational ruled surface can be found in [GH78, Chapter 4].

Let z_1 be an inhomogeneous coordinate on $\mathbb{CP}^1$ and consider the standard Kähler metric, the Fubini-Study metric

$$\omega_{FS} = \frac{dz_1 \wedge d\bar{z}_1}{(1+z_1\bar{z}_1)^2}.$$

Using the fact that $\mathcal{O}_{\mathbb{CP}^1}(-2) = K_{\mathbb{CP}^1}$ and $\mathcal{O}_{\mathbb{CP}^1}(2) = T\mathbb{CP}^1$, we have a natural Hermitian metric on $\mathcal{O}_{\mathbb{CP}^1}(n) \oplus \mathcal{O}_{\mathbb{CP}^1}$ defined as follows: Let $w \in \mathcal{O}_{\mathbb{CP}^1}(n) \oplus \mathcal{O}_{\mathbb{CP}^1}$. Then $w = (z_1, w_1(dz_1)^{-\frac{n}{2}}, w_2)$, where w_1, w_2 are coordinates in the fiber direction and $(dz_1)^{-1}$ is a section of $\mathcal{O}_{\mathbb{CP}^1}(2) = T\mathbb{P}^1$. From this, we see that $(dz_1)^{-\frac{n}{2}}$ is a section of $\mathcal{O}_{\mathbb{CP}^1}(n)$. In the fiber direction, we have

$$||w||^2 = w_1\bar{w}_2(1+z_1\bar{z}_1)^n + w_2\bar{w}_2.$$

Taking local inhomogeneous coordinates $z_2 = w_2/w_1$ yields the following form on $\mathcal{O}_{\mathbb{CP}^1}(n) \oplus \mathcal{O}_{\mathbb{CP}^1}$

$$\hat{\varphi} = \pi^*\omega_{FS} + s\sqrt{-1}\partial\bar{\partial}\log||w||^2,$$

where s is a positive real number chosen small enough so that $\hat{\varphi}$ is positive-definite, and π is the projection map $\pi : \mathcal{O}_{\mathbb{CP}^1}(n) \oplus \mathcal{O}_{\mathbb{CP}^1} \to \mathbb{CP}^1$.

Projectivizing the fibers of this direct sum of line bundles yields a closed form φ_s. Using the fact that the associated $(1,1)$-form of the Fubini-Study metric is $\sqrt{-1}\partial\bar{\partial}\log|z|^2 = \sqrt{-1}\partial\bar{\partial}\log(1+z_1\bar{z}_1)$ and that $z_2 = w_2/w_1$, we have

$$\varphi_s = \sqrt{-1}\partial\bar{\partial}\left(\log(1+z_1\bar{z}_1) + s\log((1+z_1\bar{z}_1)^n + z_2\bar{z}_2)\right), \tag{4.2}$$

which is globally well-defined on $\mathbb{F}_n$. This is the form of the metric which we will use in this section. It should be remarked that the metric in (4.2) is Kähler. When

$s \in \mathbb{Q}^+$, the metric is also a Hodge metric. Furthermore, φ_s is positive-definite when $s < \frac{1}{n^2}$. In fact, Hitchin used φ_s to prove the following theorem:

Theorem 4.4 ([Hit75])**.** *For all $n \geq 0$, $\mathbb{F}_n$ admits [Hodge] metrics of positive holomorphic sectional curvature.*

Before proving Theorem 4.4, we make the following remarks:

Remark 4.5. Recall that $SU(2)$ acts on $\mathbb{CP}^1$ as an isometry of the Fubini-Study metric, preserves the fiber metric when lifted to the vector bundle, and acts transitively on $\mathbb{CP}^1$. Hence, without lost of generality, we can simplify our calculations by restricting our computations along one fiber—say $z_1 = 0$.

Remark 4.6. In [Hit75], the $R_{i\bar{j}k\bar{l}}$ are expressed in terms of a unitary frame field. The proof of Theorem 4.4 presented below involves the curvature tensors in terms of the frame $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\}$ with respect to the coordinates mentioned above. We use this frame since it lends itself to the methods utilized when computing the pinching constant in this section.

Proof. Let $G := \log(1 + z_1\bar{z}_1) + s\log((1 + z_1\bar{z}_1)^n + z_2\bar{z}_2)$. Thus

$$\varphi_s = \sqrt{-1}\partial\bar{\partial}G = \sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j.$$

Direct computation yields

$$(g_{i\bar{j}}) = \begin{pmatrix} \frac{1+z_2\bar{z}_2+sn}{1+z_2\bar{z}_2} & 0 \\ 0 & \frac{s}{(1+z_2\bar{z}_2)^2} \end{pmatrix} \quad \text{and} \quad (g^{i\bar{j}}) = \begin{pmatrix} \frac{1+z_2\bar{z}_2}{1+z_2\bar{z}_2+sn} & 0 \\ 0 & \frac{(1+z_2\bar{z}_2)^2}{s} \end{pmatrix}.$$

From (2.6), the components of the curvature tensor are

$$R_{1\bar{1}1\bar{1}} = \frac{2(-n^2 s z_2\bar{z}_2 + (1+z_2\bar{z}_2)^2 + n(s + s z_2\bar{z}_2))}{(1+z_2\bar{z}_2)^2} \tag{4.3}$$

$$R_{1\bar{1}2\bar{2}} = \frac{ns(1+ns-z_2^2\bar{z}_2^2)}{(1+z_2\bar{z}_2)^3(1+ns+z_2\bar{z}_2)}, \tag{4.4}$$

$$R_{2\bar{2}2\bar{2}} = \frac{2s}{(1+z_2\bar{z}_2)^4}, \tag{4.5}$$

while the other terms, except those obtained from symmetry, are equal to zero. Let $\xi \in T_{(0,z_2)}\mathbb{F}_n$ be an arbitrary unit tangent vector such that $\xi = \xi_1 \frac{\partial}{\partial z_1} + \xi_2 \frac{\partial}{\partial z_2}$. When we substitute the $R_{i\bar{j}k\bar{l}}$ into the formula for holomorphic sectional curvature we obtain

$$K(\xi) = \frac{4}{(1+z_2\bar{z}_2)^4}(\xi_1\bar{\xi}_1)^2(1+z_2\bar{z}_2) + \frac{4s}{(1+z_2\bar{z}_2)^4}\left((\xi_1\bar{\xi}_1)^2(1+z_2\bar{z}_2)^3 + (\xi_2\bar{\xi}_2)^2\right) + \frac{4s}{(1+z_2\bar{z}_2)^4}\left(\frac{-(\xi_1\bar{\xi}_1)^2 n^2 z_2\bar{z}_2(1+z_2\bar{z}_2)^2 + 2n(\xi_1\bar{\xi}_1)(\xi_2\bar{\xi}_2)(1+z_2\bar{z}_2)(1+ns-z_2^2\bar{z}_2^2)}{1+ns+z_2\bar{z}_2}\right).$$

We observe that when $\xi \neq 0$, we have $K(\xi) > 0$ since the first term is positive and we are letting s be sufficiently small—particularly $s < \frac{1}{n^2}$. Thus, $\mathbb{F}_n$ has a metric which admits positive holomorphic sectional curvature for all $n \geq 0$. □

This result of positivity may be considered surprising, as the $\mathbb{F}_n$ do not carry metrics of positive Ricci curvature for certain values of n. To show this, we will use some results discussed in Section 2.4.

Proposition 4.7. *For $n \geq 2$, $\mathbb{F}_n$ does not admit a metric of positive Ricci curvature.*

Proof. By Theorem 2.37, it suffices to show that $-K_{\mathbb{F}_n}$ is not ample for $n \geq 2$. By Nakai's Criterion in Chapter 2, a line bundle is ample if its self-intersection number is positive and its intersection with *any* irreducible curve is positive. Consider the curve E_∞ on $\mathbb{F}_n$. Using the Adjunction Formula II found in [GH78, Chapter 1], we have $K_{E_\infty} = K_{\mathbb{F}_n}|_{E_\infty} \otimes [E_\infty]|_{E_\infty}$. Then by the self-intersection number of E_∞ from (4.1), we have

$$\begin{aligned} \deg K_{E_\infty} &= \deg K_{\mathbb{F}_n}|_{E_\infty} + \deg[E_\infty]|_{E_\infty} \\ &= \deg K_{\mathbb{F}_n}|_{E_\infty} + E_\infty.E_\infty \\ &= \deg K_{\mathbb{F}_n}|_{E_\infty} + (-n) \\ &= K_{\mathbb{F}_n}.E_\infty - n. \end{aligned}$$

Since $\deg K_{E_\infty} = 2g-2$ and $g = 0$, we know $\deg K_{E_\infty} = -2$. Hence, $-2 = K_{\mathbb{F}_n}.E_\infty - n$, and

$$-K_{\mathbb{F}_n}.E_\infty = 2 - n.$$

When $n \geq 2$, we see that the intersection number is not positive. Thus, $-K_{\mathbb{F}_n}$ is not ample for $n \geq 2$. □

Due to the compactness of $\mathbb{F}_n$, we know that a minimum value and maximum value of the holomorphic sectional curvature exist. Although Hitchin proved $\mathbb{F}_n$ has positive holomorphic sectional curvature, his proof did not yield any pinching constants. With this motivation, we have the following pinching result:

Theorem 4.8 ([ACH15])**.** *Let $\mathbb{F}_n$, $n \in \{1, 2, 3, \ldots\}$, be the n-th Hirzebruch surface. Then there exists a Hodge metric on $\mathbb{F}_n$ whose holomorphic sectional curvature is $\frac{1}{(1+2n)^2}$-pinched.*

Proof. We first consider the case when $n \geq 2$. Take

$$\varphi_s = \sqrt{-1}\partial\bar{\partial}[\log(1 + z_1\bar{z_1}) + s\log((1 + z_1\bar{z_1})^n + z_2\bar{z_2})].$$

From

$$(g_{i\bar{j}}) = \begin{pmatrix} \frac{1+z_2\bar{z}_2+sn}{1+z_2\bar{z}_2} & 0 \\ 0 & \frac{s}{(1+z_2\bar{z}_2)^2} \end{pmatrix},$$

we see that an orthonormal basis for $T_{(0,z_2)}\mathbb{F}_n$ is $\{\vec{u_1}, \vec{u_2}\}$, where

$$\vec{u_1} := \sqrt{\frac{1+z_2\bar{z}_2}{1+z_2\bar{z}_2+ns}} \cdot \frac{\partial}{\partial z_1}, \quad \text{and} \quad \vec{u_2} := \frac{1+z_2\bar{z}_2}{\sqrt{s}} \cdot \frac{\partial}{\partial z_2}.$$

Therefore, an arbitrary unit tangent vector, $\xi \in T_{(0,z_2)}\mathbb{F}_n$, can be written as

$$\xi = c_1\sqrt{\frac{1+z_2\bar{z}_2}{1+z_2\bar{z}_2+ns}} \cdot \frac{\partial}{\partial z_1} + c_2\frac{1+z_2\bar{z}_2}{\sqrt{s}} \cdot \frac{\partial}{\partial z_2},$$

where $c_1, c_2 \in \mathbb{C}$ such that $|c_1|^2 + |c_2|^2 = 1$. Define the following:

$$\xi_1 := c_1\sqrt{\frac{1+z_2\bar{z}_2}{1+z_2\bar{z}_2+ns}}, \quad \text{and} \quad \xi_2 := c_2\frac{1+z_2\bar{z}_2}{\sqrt{s}}.$$

Substituting the values of $R_{i\bar{j}k\bar{l}}$ from (4.3), (4.4), and (4.5), and the values of ξ_1 and ξ_2 into the definition of holomorphic sectional curvature gives us

$$\begin{aligned} K(\xi) &= 2R_{1111}\xi_1\bar{\xi}_1\xi_1\bar{\xi}_1 + 8R_{1122}\xi_1\bar{\xi}_1\xi_2\bar{\xi}_2 + 2R_{2222}\xi_2\bar{\xi}_2\xi_2\bar{\xi}_2 \\ &= \frac{4((1+z_2\bar{z}_2)^2 + ns(1+z_2\bar{z}_2 - nz_2\bar{z}_2))}{(1+z_2\bar{z}_2+ns)^2}|c_1|^4 \\ &+ \frac{8n(1+ns-z_2^2\bar{z}_2^2)}{(1+z_2\bar{z}_2+ns)^2}|c_1|^2|c_2|^2 + \frac{4}{s}|c_2|^4. \end{aligned} \tag{4.6}$$

Since this expression of $K(\xi)$ only depends on the modulus squared of z_2, we let $r := z_2\bar{z}_2$. Also, because $|c_1|^2$ and $|c_2|^2$ are nonnegative real numbers, we can let

$a := |c_1|^2$ and $b := |c_2|^2$. Let

$$\alpha := \frac{4((1+r)^2 + ns(1+r-nr))}{(1+r+ns)^2}, \quad \beta := \frac{8n(1+ns-r^2)}{(1+r+ns)^2}, \quad \gamma := \frac{4}{s}$$

be the coefficients in (4.6).

With these substitutions, and for fixed values of r and s, the holomorphic sectional curvature takes the form of the following degree 2 homogeneous polynomial in a and b with real coefficients:

$$K_{r,s}(a,b) = \alpha a^2 + \beta ab + \gamma b^2. \tag{4.7}$$

This is the function we want to maximize and minimize in order to find the pinching constant for φ_s, subject to the constraint $a + b = 1$. We find the extrema utilizing the method of Lagrange Multipliers. Keeping r fixed, we have the equations

$$\frac{\partial}{\partial a} K_{r,s}(a,b) = \lambda, \ \frac{\partial}{\partial b} K_{r,s}(a,b) = \lambda, \ a + b - 1 = 0.$$

Solving this system of equations for a and b yields a unique stationary solution in the interior

$$a_0 = \frac{2\gamma - \beta}{2(\gamma - \beta + \alpha)} = \frac{(1+r)(1+ns)}{1 + s - (-1+n)ns^2 + r(1+s+2ns)}$$
$$b_0 = \frac{2\alpha - \beta}{2(\gamma - \beta + \alpha)} = \frac{s(-1+n-r-nr-ns+n^2 s)}{-1-r-s-rs-2nrs-ns^2+n^2 s^2}.$$

We then observe the holomorphic sectional curvature in three cases:

1. At the interior stationary point (a_0, b_0)

2. At the boundary point $(0, 1)$

3. At the boundary point $(1,0)$.

In each case, after substituting the values of (a,b) into (4.7) with s fixed, we obtain the following functions, some of which depend on r:

1. $f_s : [0,\infty) \to \mathbb{R}^+$, where

$$\begin{aligned} f_s(r) &:= K_{r,s}(a_0,b_0) \\ &= \frac{4\left(3r^2(1+ns) + +3r(1+ns)^2 - r^3(-1+n^2s) - (1+ns)^2(-1-ns+n^2s)\right)}{(1+r+ns)^2(1+s-(-1+n)ns^2+r(1+s+2ns))} \end{aligned}$$

2. $K_{r,s}(0,1) = \frac{4}{s}$

3. $h_s : [0,\infty) \to \mathbb{R}^+$, where

$$h_s(r) := K_{r,s}(1,0) = \frac{4((1+r)^2 + ns(1+r-nr))}{(1+r+ns)^2}.$$

We then determine the extrema of each function in the interior $(0,\infty)$ and at the endpoints 0 and ∞.

1. For $f_s(r) := K_{r,s}(a_0,b_0)$, we see that $f_s'(r) = 0$ if and only if $r = -1$ or if $r = \frac{(n-1)(1+ns)}{1+n}$. Since $r = -1 \notin (0,\infty)$, we only regard the second critical point since it is inside $(0,\infty)$ for $n \geq 2$. Note that

$$f_s\left(\frac{(n-1)(1+ns)}{1+n}\right) = \frac{4-s(n-1)^2}{1+ns}.$$

At the endpoints of the interval $[0,\infty)$, we see that

$$f_s(0) = \frac{4(1+ns-n^2s)}{1+s-(n-1)ns^2}, \quad \text{and} \quad \lim_{r\to\infty} f_s(r) = \frac{4-4n^2s}{1+s+2ns}.$$

The latter expression makes it clear that we need to choose $s < \frac{1}{n^2}$ in order to obtain positive holomorphic sectional curvature. Furthermore, for $s < \frac{1}{n^2}$,

$$\frac{4(1+ns-n^2s)}{1+s-(n-1)ns^2} - \frac{4-4n^2s}{1+s+2ns} = \frac{4s(3n-s(2n^3-3n^2)-s^2(n^4-n^3))}{(1+s+2ns)(1+s(1-s(n^2-n)))} > 0,$$

and

$$\frac{4-s(n-1)^2}{1+ns} - \frac{4(1+ns-n^2s)}{1+s-(n-1)ns^2} = -\frac{s(n-1)^2(3+s(n-1))}{(-1+s(n-1))(1+ns)} > 0.$$

Thus, we have

$$\frac{4-s(n-1)^2}{1+ns} > \frac{4(1+ns-n^2s)}{1+s-(n-1)ns^2} > \frac{4-4n^2s}{1+s+2ns},$$

and

$$\frac{4-4n^2s}{1+s+2ns} \leq K_{r,s}(a_0, b_0) \leq \frac{4-s(n-1)^2}{1+ns}.$$

2. For $K_{r,s}(0,1) = \frac{4}{s}$, the curvature is independent of r and is constant.

3. For $h_s(r) := K_{r,s}(1,0)$, we see that $h_s'(r) = 0$ if and only if $r = \frac{(n-1)(1+ns)}{1+n}$, which is inside $(0,\infty)$ for $n \geq 2$. We observe that $h_s(r)$ and $f_s(r)$ have a common critical point, although there does not appear to be a clear geometric reason for this coincidence. Note that

$$h_s\left(\frac{(n-1)(1+ns)}{1+n}\right) = \frac{4-s(n-1)^2}{1+ns}.$$

At the endpoints, we have

$$h_s(0) = \frac{4}{1+ns}, \quad \text{and} \quad \lim_{r\to\infty} h_s(r) = 4.$$

By comparing these values while taking $s < \frac{1}{n^2}$, we arrive at

$$\frac{4 - s(n-1)^2}{1+ns} \leq K_{r,s}(1,0) \leq 4.$$

When we compare the infima from each case, we have

$$\frac{4-4n^2 s}{1+s+2ns} \leq \frac{4-s(n-1)^2}{1+ns} \leq \frac{4}{s}.$$

Comparing the suprema from each case, we have

$$\frac{4-s(n-1)^2}{1+ns} \leq 4 \leq \frac{4}{s}.$$

Thus, the smallest value attained for the holomorphic sectional curvature is $\frac{4-4n^2 s}{1+s+2ns}$ and the largest value is $\frac{4}{s}$.

To find the best value of s with the best pinching constant, we define the function

$$p : \left(0, \frac{1}{n^2}\right) \to (0,1), \quad p(s) := \frac{\min_\xi K_s(\xi)}{\max_\xi K_s(\xi)} = \frac{\frac{4-4n^2 s}{1+s+2ns}}{\frac{4}{s}} = \frac{s(1-n^2 s)}{1+s+2ns},$$

where the minimum and maximum are taken over all (unit) tangent vectors across the entire manifold. This is the function we want to maximize.

We see that $p'(s) = 0$ if and only if $s = -\frac{1}{n}$ or if $s = \frac{1}{2n^2+n}$. Since the first critical point is not inside $\left(0, \frac{1}{n^2}\right)$, we only consider the latter value. Calculus-style computation tells us that p has a global maximum at $s = \frac{1}{2n^2+n}$. This gives us the optimal pinching of

$$p\left(\frac{1}{2n^2+n}\right) = \frac{1}{(1+2n)^2}.$$

Next, we consider the case when $n = 1$. When $n = 1$, the functions f_s and h_s have their critical points at $r = 0$. Using a very similar argument as above, we see that the pinching constant is equal to $\frac{1}{9}$ for $s = \frac{1}{3}$. □

4.2 Products of Manifolds with Positive Curvature

In the case of the 0-th Hirzebruch surface, we have that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. With the Fubini-Study product metric, it was computed that the holomorphic sectional curvature is $\frac{1}{2}$-pinched. Additionally, for a general product of projective spaces, $\mathbb{P}^n \times \mathbb{P}^m$ for $m, n \in \{1, 2, ...\}$, the pinching constant of the holomorphic sectional curvature is also $\frac{1}{2}$. In particular, it was observed that $\frac{1}{2} = \frac{1}{(1+1)}$, where 1 is equal to the pinching constant of $\mathbb{CP}^n$ (since $\mathbb{CP}^n$ has constant holomorphic sectional curvature equal to 4).

For general products of Hermitian manifolds of positive holomorphic sectional curvature, we have the following result:

Theorem 4.9 ([ACH15]). *Let M and N be Hermitian manifolds whose positive holomorphic sectional curvatures are c_M- and c_N-pinched, respectively, and satisfy*

$$kc_M \leq K_M \leq k \quad \textit{and} \quad kc_N \leq K_N \leq k$$

for a constant $k > 0$. Then the holomorphic sectional curvature K of the product *metric on* $M \times N$ *satisfies*

$$k\frac{c_M c_N}{c_M + c_N} \leq K \leq k$$

and is $\frac{c_M c_N}{c_M + c_N}$*-pinched.*

Proof. Let $g = \sum_{i,j=1}^{m} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$, and $h = \sum_{i,j=m+1}^{m+n} h_{i\bar{j}} dz_i \otimes d\bar{z}_j$ be Hermitian metrics on M and N, respectively, each with positive holomorphic sectional curvature. Then

$$\sum_{i,j=1}^{m} g_{i\bar{j}} dz_i \otimes d\bar{z}_j + \sum_{i,j=m+1}^{m+n} h_{i\bar{j}} dz_i \otimes d\bar{z}_j$$

gives the product metric in a neighborhood of $(P, Q) \in M \times N$. It should be observed that the $g_{i\bar{j}}$ are functions of only $z_1, \ldots, z_m$, and the $h_{i\bar{j}}$ are functions of only $z_{m+1}, \ldots, z_{m+n}$. Using (2.6) we obtain

$$R_{i\bar{j}k\bar{l}} = \begin{cases} -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{m} g^{q\bar{p}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{j}}}{\partial \bar{z}_l}, & 1 \leq i,j,k,l \leq m \\ \\ -\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=m+1}^{m+n} h^{q\bar{p}} \frac{\partial h_{i\bar{p}}}{\partial z_k} \frac{\partial h_{q\bar{j}}}{\partial \bar{z}_l}, & m+1 \leq i,j,k,l \leq m+n \\ \\ 0, & \text{otherwise.} \end{cases}$$

Let $\xi = \sum_{i=1}^{m+n} \xi_i \frac{\partial}{\partial z_i}$ be a unit tangent vector in $T_{(P,Q)}(M \times N)$. Then the holomorphic sectional curvature on $M \times N$ along ξ is

$$\begin{aligned} K(\xi) = 2 \sum_{i,j,k,l=1}^{m} \left(-\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{m} g^{q\bar{p}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{j}}}{\partial \bar{z}_l} \right) \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l \\ + 2 \sum_{i,j,k,l=m+1}^{m+n} \left(-\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=m+1}^{m+n} h^{q\bar{p}} \frac{\partial h_{i\bar{p}}}{\partial z_k} \frac{\partial h_{q\bar{j}}}{\partial \bar{z}_l} \right) \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l. \end{aligned}$$

Because $-\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{m} g^{q\bar{p}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{j}}}{\partial \bar{z}_l}$ and $-\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=m+1}^{m+n} h^{q\bar{p}} \frac{\partial h_{i\bar{p}}}{\partial z_k} \frac{\partial h_{q\bar{j}}}{\partial \bar{z}_l}$ are equal to the components of the curvature tensor on M and N, respectively, the two sums on the right-hand side above are just the numerators of the holomorphic sectional curvatures on M and N with respect to the tangent vectors $(\xi_1, \dots, \xi_m) \in T_P M$ and $(\xi_{m+1}, \dots, \xi_{m+n}) \in T_Q N$. Because both curvatures are assumed to be positive, we can conclude that $K(\xi) > 0$.

In order to find the pinching constant, we need to take into consideration the (nonzero) norms of $(\xi_1, \dots, \xi_m) \in T_P M$ and $(\xi_{m+1}, \dots, \xi_{m+n}) \in T_Q N$ with respect to their respective metrics in the two spaces. We do this as follows:

$$
\begin{aligned}
K(\xi) &= \sum_{i,k,j,l=1}^{m} 2R_{i\bar{j}k\bar{l}} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l + \sum_{i,k,j,l=m+1}^{m+n} 2R_{i\bar{j}k\bar{l}} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l \\
&= \frac{\sum_{i,k,j,l=1}^{m} 2R_{i\bar{j}k\bar{l}} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l}{\sum_{i,j,k,l=1}^{m} g_{i\bar{j}} g_{k\bar{l}} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l} \cdot \sum_{i,j,k,l=1}^{m} g_{i\bar{j}} g_{k\bar{l}} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l \\
&\quad + \frac{\sum_{i,k,j,l=m+1}^{m+n} 2R_{i\bar{j}k\bar{l}} \xi \bar{\xi}_j \xi_k \bar{\xi}_l}{\sum_{i,j,k,l=m+1}^{m+n} h_{i\bar{j}} h_{k\bar{l}} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l} \cdot \sum_{i,j,k,l=m+1}^{m+n} h_{i\bar{j}} h_{k\bar{l}} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l \\
&= K_M \cdot y^2 + K_N \cdot (1-y)^2,
\end{aligned}
$$

where K_M is the holomorphic sectional curvature of M in the direction of $(\xi_1, \dots, \xi_m)$, K_N the holomorphic sectional curvature of N in the direction of $(\xi_{m+1}, \dots, \xi_{m+n})$, and $y := \sum_{i,j=1}^{m} g_{i\bar{j}} \xi_i \bar{\xi}_j$. Since ξ is a unit tangent vector in $T_{(P,Q)}(M \times N)$, we have that

$$
\sum_{i,j=1}^{m} g_{i\bar{j}} \xi_i \bar{\xi}_j + \sum_{i,j=m+1}^{m+n} h_{i\bar{j}} \xi_i \bar{\xi}_j = 1.
$$

Hence,

$$\sum_{i,j=m+1}^{m+n} h_{i\bar{j}}\xi_i\bar{\xi}_j = 1 - \sum_{i,j=1}^{m} g_{i\bar{j}}\xi_i\bar{\xi}_j = 1 - y.$$

Furthermore, the assumption

$$kc_M \leq K_M \leq k \quad \text{and} \quad kc_N \leq K_N \leq k$$

provides the following inequality:

$$F(y) := kc_M y^2 + kc_N(1-y)^2 \leq K_M y^2 + K_N(1-y)^2 \leq ky^2 + k(1-y)^2 =: \widetilde{F}(y).$$

Finally, calculus-style computation yields

$$\min_{0\leq y\leq 1} F(y) = k\frac{c_M c_N}{c_M + c_N} \quad \text{and} \quad \max_{0\leq y\leq 1} \widetilde{F}(y) = k.$$

In particular, $k\frac{c_M c_N}{c_M + c_N} \leq K(\xi) \leq k$, and the pinching constant for the holomorphic sectional curvature on the product manifold is obtained as

$$c_{M\times N} = \frac{\inf_\xi K(\xi)}{\sup_\xi K(\xi)} = \frac{c_M c_N}{c_M + c_N}.$$

□

This pinching result may come as surprising due to the following conjecture in Riemannian geometry:

Conjecture 4.10. *(The Hopf Conjecture) The product of two real 2-spheres $S^2 \times S^2$ does not admit a Riemannian metric of positive sectional curvature.*

Some remarks on this conjecture can be found in [Wil07]. Since the conjecture is considering (full) Riemannian sectional curvature and we are only considering sectional curvature along complex real 2-planes, the conjecture does not contradict with Theorem 4.9. Hence, Theorem 4.9 shows that holomorphic sectional curvature is actually more "well-behaved" than Riemannian sectional curvature.

Bibliography

[ACH15] A. Alvarez, A. Chaturvedi, and G. Heier. Optimal pinching for the holomorphic sectional curvature of Hitchin's metrics on Hirzebruch surfaces. *Contemp. Math.* **654**, 133–142, 2015.

[AHZ15] A. Alvarez, G. Heier, and F. Zheng. An example of a compact Kähler manifold with positive holomorphic sectional curvature. In preparation.

[Bal06] W. Ballmann. *Lectures on Kähler manifolds.* European Mathematical Society, 2006.

[Ber66] M. Berger. Sur les variétés d'Einstein compactes. *Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (Namur, 1965)*, 35–55, 1966.

[Bir09] G.D. Birkhoff. Singular points of ordinary linear differential equations. *Trans. Amer. Math. Soc.* **10**(4), 436–470, 1909.

[BS08] S. Brendle and R. Schoen. Classification of manifolds with weakly 1/4-pinched curvatures. *Acta Math.* **200**(1), 1–13, 2008.

[BS09] S. Brendle and R. Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.* **22**(4), 287–307, 2009.

[Che89] C.K. Cheung. Hermitian metrics of negative holomorphic sectional curvature on some hyperbolic manifolds. *Math Z.* **201**, 105–119, 1989.

[Che77] B.-Y. Chen. Extrinsic spheres in Kähler manifolds, II. *Michigan Math. J.* **24**(1), 97–102, 1977.

[GH78] P. Griffiths and J. Harris. *Principles of algebraic geometry.* John Wiley & Sons, New York, 1978.

[Gri69] P. A. Griffiths. Hermitian differential geometry, Chern classes, and positive vector bundles. *Global Analysis (Papers in honor of K. Kodaira)*, 185–251, 1969.

[Gro57] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.* **79**, 121–138, 1957.

[Hor64] G. Horrocks. Vector bundles on the punctured spectrum of a local ring. *Proc. London Math. Soc.* **14**(3), 689–713, 1964.

[Hit75] N. Hitchin. On the curvature of rational surfaces. *Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973)*, Amer. Math. Soc., 65–80, 1975.

[HLW10] G. Heier, S. Lu, and B. Wong. On the canonical line bundle and negative holomorphic sectional curvature. *Math. Res. Lett.* **17**(6), 1101–1110, 2010.

[HLW15] G. Heier, S. Lu, and B. Wong. Kähler manifolds of semi-negative holomorphic sectional curvature. arXiv:1403.4210, to appear in *J. Differential Geom.*, 2015.

[HW12] G. Heier and B. Wong. Scalar curvature and uniruledness on projective manifolds. *Comm. Anal. Geom.* **20**(4), 751–764, 2012.

[HW15] G. Heier and B. Wong. On projective Kähler manifolds of partially positive curvature and rational connectedness. arXiv:1509.02149, 2015.

[Kle66] S. Kleiman. Toward a numerical theory of ampleness. *Ann. of Math.* **84**(2), 293–344, 1966.

[KN69] S. Kobayashi and K. Nomizu. *Foundations of differential geometry, Vol. II.* Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II. Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[Kob70] S. Kobayashi. *Hyperbolic manifolds and holomorphic mappings.* Marcel Dekker, New York, 1970.

[Laz04] R. Lazarsfeld. *Positivity in Algebraic Geometry, Vol. 1.* Springer-Verlag, Berlin, Heidelberg, 2004.

[Moi61] B. Moishezon. Projective imbeddings of algebraic manifolds. *Dokl. Akad. Nauk SSSR* **141**, 555–557, 1961.

[Moi62] B. Moishezon. Remarks on projective imbeddings of algebraic manifolds. *Dokl. Akad. Nauk SSSR* **145**, 996–999, 1962.

[Nak60] Y. Nakai. Non-degenerate divisors on an algebraic surface. *J. Sci. Hiroshima Univ. Ser. A* **24**, 1–6, 1960.

[Nak63] Y. Nakai. A criterion of an ample sheaf on a projective scheme. *Amer. J. Math.* **85**, 14–26, 1963.

[SS85] K. Sekigawa and T. Sato. Nearly Kähler manifolds with positive holomorphic sectional curvature. *Kodai Math. J.* **8**, 139–156, 1985.

[Shi71] B. Shiffman. Extension of holomorphic maps into hermitian manifolds. *Math. Ann.* **194**, 249–258, 1971.

[SY10] R. Schoen and S.-T. Yau. *Lectures on differential geometry.* International Press, Boston, MA, 2010.

[TY15] V. Tosatti and X. Yang. An extension of a theorem of Wu-Yau. arXiv:1506.01145, 2015.

[Wil07] B. Wilking. Nonnegatively and postively curved manifolds. arXiv:0707.3091v1, 2007.

[Wu73] H. Wu. A remark on holomorphic sectional curvature. *Indiana Univ. Math. J.* **22**, 1103–1108, 1972/73.

[WY15] D. Wu and S.-T. Yau. Negative holomorphic curvature and positive canonical bundle. arXiv:1505.05802, to appear in *Invent. Math.*, 2015.

[Yau78] S.T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation, I. *Comm. Pure Appl. Math.* **31**, 339–441, 1978.

[Yan15] X. Yang. Hermitian manifolds with semi-positive holomorphic sectional curvature. arXiv:1412.5252, 2015.

[Zhe00] F. Zheng. *Complex differential geometry.* AMS/IP Studies in Advanced Mathematics, **18**. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2000.