HEDGING WITH EUROPEAN DOUBLE BARRIER BASKET OPTIONS AS A CONTROL CONSTRAINED OPTIMAL CONTROL PROBLEM

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Daqian Li

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Abstract

In finance, hedging strategies are used to safeguard portfolios against risk associated with financial derivatives such as options. For an option with an underlying asset, the risk can be measured in terms of the so-called Greeks. In particular, the derivative of the option price with respect to the value of the asset is referred to as the Delta. An alternative to optimize hedges for options is to optimize options for hedging. Here, we are concerned with European double barrier basket options with multiple cash settlements. The cash settlements are considered as controls and the objective is to choose the controls such that the Delta is as close to a constant as possible. This amounts to the solution of a control constrained optimal control problem for the multidimensional Black Scholes equation featuring Dirichlet boundary control and final time control. We prove existence and uniqueness of the optimal control and derive the first order necessary optimality conditions in terms of the state, the adjoint state, and the control. The numerical solution is based on a discretization in space by P1 conforming finite elements with respect to a simplicial triangulation of the spatial domain and a further discretization in time by the implicit Euler scheme with respect to a partition of the time interval. The fully discretized optimal control problem is then solved by a projected gradient method with Armijo line search. Numerical results are given to illustrate the performance of the suggested approach.

Contents

Chapter 1

Introduction

Options that are different from plain vanilla American or European call or put options are commonly referred to as exotic options (cf., e.g., [24, 34, 47]). Among the exotic options, those of single or double barrier type are of particular interest. Such options become effective (knock-in options) or expire (knock-out options) as soon as the value of the underlying asset hits some prespecified upper and/or lower barrier. The valuation of a single barrier option with one underlying asset has been studied first by Merton [32] and subsequently investigated in [9, 12, 37, 39]. As far as barrier options with more than one underlying asset are concerned, one of the first contributions was [20] dealing with barrier options on a single stock where the barrier is determined by another asset. Valuation formulas for barrier options on a basket have been derived later in [26, 46]. Hedging techniques for barrier options have been considered by different approaches including static hedging [10, 11, 36], the partial differential equation (PDE) formulation [2, 14, 29, 33, 36, 40], and stochastic optimization [18, 30, 31].

In this thesis, we will study an optimal control approach for hedging barrier options with multiple cash settlements at the option's expiration [6]. The thesis is organized as follows: In chapter 2, we begin with the basic principles of the theory of option pricing. We introduce plain vanilla European/American options and exotic options focusing on European Double Barrier Basket Options (section 2.1) followed by a brief discussion of various types of traders in section 2.2. Section 2.4 is devoted to the well-known Black-Scholes-Merton model for the evaluation of the fair price of a European option for one underlying asset, whereas section 2.5 addresses the multidimensional case in terms of a basket of assets. The variational (weak) formulation of the final time/boundary value problem for the Black-Scholes equation for a European put option on two underlyings is given in section 2.6 on the basis of weighted Sobolev spaces. In chapter 3, we will be concerned with hedging strategies with emphasis on Delta hedging. For that purpose, we will introduce Greeks in section 3.1 and futures contracts in section 3.2. As standard hedging instruments, we consider Delta hedging with options and futures contracts in sections 3.3 and 3.4. A feasible alternative is Delta hedging with European Double Barrier Options which will be illustrated in section 3.5. In chapter 4, following the exposition in [22], we consider hedging with European double barrier basket call options on two underlying assets featuring a certain number of cash settlements at predetermined values of the underlying assets between the strike and the upper barrier (section 4.1). The cash settlements are interpreted as bilaterally constrained control variables that have to be chosen in such a way that the Delta is as close to a prespecified constant Delta as possible leading to a tracking type objective functional. We are thus faced with the solution of a control constrained optimal control problem for the two-dimensional Black-Scholes equation in the space-time domain $Q := \Omega \times (0,T), T > 0$, where Ω is a trapezoidal domain in $\mathbb{R}^2$ determined by the lower and upper barriers K_{min} and K_{max}. In particular, the cash settlement at the upper barrier represents a Dirichlet boundary control, whereas the other cash settlements occur as a final time control vector.

As a particular feature, the Dirichlet boundary conditions on the boundaries parallel to the coordinate axes are given by the solution of associated one-dimensional Black-Scholes equations (section 4.2). In section 4.2, using a simple transformation in time, we rewrite the problem as an initial control/Dirichlet boundary control problem and consider its variational formulation in a weighted Sobolev space setting. The first order necessary optimality conditions involving adjoint states that satisfy backward in time parabolic PDEs as well as a variational inequality due to the bilateral constraints on the control will be derived in section 4.2. In section 4.3, we are concerned with the discretization of the optimal control problem. We first consider a semi-discretization in space by conforming P1 finite elements with respect to a simplicial triangulation of the computational domain. The semi-discretized control problem requires the minimization of a semi-discrete objective functional subject to systems of first order ordinary differential equations (ODEs) obtained by the finite element approximation in space and subject to the bilateral constraints on the controls. It represents a control constrained initial control problem for the corresponding systems of first order ODEs in terms of the associated mass and stiffness matrices as well as the input matrices expressing the input from the semi-discretized boundary controls at the upper barrier. The semi-discrete optimality system reflects the intrinsic relationships between the states, the adjoint states, and the controls. A further discretization in time with respect to a partition of the time interval gives rise to a fully discrete optimization problem. Chapter 5.2 contains the numerical solution of the fully discrete optimal control problem by a projected gradient method with line search as well as a documentation of representative numerical results. The final chapter 6 summarizes the basic results of this thesis and gives an outlook on possible future work.

Chapter 2

Pricing Of Options

In this chapter, we introduce the basic principles of the theory of option pricing. In section 2.1, we discuss plain vanilla European and American options as well as exotic options with emphasis on European Double Barrier Basket Options. Different types of traders are addressed in section 2.2. Then, we present the well-known Black-Scholes-Merton model for the evaluation of the fair price of a European option both for one underlying asset (section 2.4) and for a basket of assets (section 2.5) followed by the variational formulation of the final time boundary value problem for the Black-Scholes equation (section 2.6). In section 3, we will be concerned with hedging strategies. After a brief introduction to Greeks and futures contracts in subsections 3.1 and 3.2, we will first consider Delta hedging with options and futures contracts (subsections 3.3 and 3.4) as standard hedging tools and then concentrate on Delta hedging with European Double Barrier Options as an attractive alternative which combines the advantages of hedging with options and futures contracts (subsection 3.5).

2.1 Types Of Options

An option(cf., e.g., [1, 4, 15]) is the right, but not the obligation, to buy or sell an asset at a fixed price at the end or within a prespecified period of time. It is a financial instrument that allows to make a bet on rising or falling values of an underlying asset. The underlying asset typically is a stock, or a parcel of shares of a company. An option is a contract between two parties about trading the asset at a certain future time. One party is the writer, often a bank, who fixes the terms of the contract and sells the option. The other party is the holder who purchases the option paying the market price which is called premium.

Several factors have an effect on the price of an option: the initial price S_0 of the underlying asset at the initial time $t = 0$, the maturity (expiry) date T, the fixed strike price K, the volatility of the underlying asset, and the (fixed) interest rate r.

There are various option types. A European (American) Vanilla Option is a contract which gives its owner the right to buy (Call) or sell (Put) a certain number of shares of the underlying asset at the strike price K until or at the maturity date T. The act of conducting the transaction is referred to as exercising the option. We call the option Vanilla, because it is a standard option type. European options can only be exercised at the expiry date T, whereas American options can be exercised any time T.

One is interested in the value of an option $y = y(S_t; t)$ (or $P(S_t; t)$ and $C(S_t; t)$ for Put/Call) depending on the spot price S_t for all $t \leq T$. Pricing a European Vanilla option at maturity goes as follows: An owner of a European Put only exercises his right to sell the stock, if the spot price S_T at the expiry date T is less than the fixed strike price K. Afterwards, he will buy the stock immediately. This leads to the payoff at maturity

$$P(S_T; T) = (K - S_T)_+ = \max(K - S_T, 0). \tag{2.1}$$

An owner of a European Call option will do the contrary: he will only exercise, if there holds $K > S_T$ and directly sell the stock. The value of the call is given by the payoff function

$$C(S_T, T) = (S_T - K)_+ = \max(S_T - K, 0). \tag{2.2}$$

For both a European Call and a European Put, the payoff function is illustrated in Figure 2.1 below in case of a strike $K = 50$.

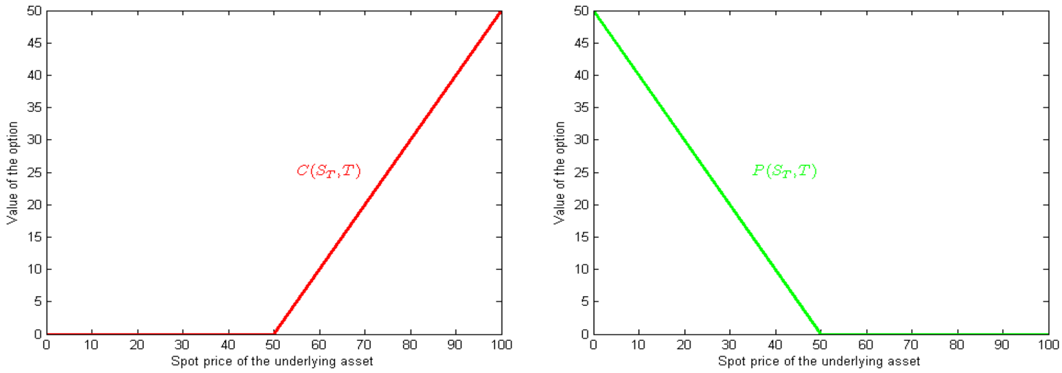

Figure 2.1: Payoff for a European Call (left) and a European Put (right).

Other than the mentioned plain vanilla options are the so-called exotic options(cf., e.g., [24, 34, 47]). These nonstandard options created by financial engineers are mostly traded at over-the-counter markets. For instance, it is possible to restrict the early exercise to certain dates as well as changing the strike price during the life of the option. Many more modifications of the option structure are possible. We are interested in European double barrier basket options. For that purpose, we introduce basket and barrier options and finally combine these two types.

Basket options are a class of options which depend on an underlying which is the value of a portfolio of assets. Stock market index options, i.e., on the Dow Jones or the S&P 500, are popular examples of a basket option.

Barrier options are a class of options which yield a payoff depending on whether the price of the underlying asset reaches a predefined level or remains in a certain interval during a fixed period of time. This yields different forms of options depending on whether the payoff vanishes (knock-out option) or the payoff begins to exist (knock-in option) when a certain barrier is reached.

We are now able to define European double barrier basket options which are the subject of this thesis: a European double barrier basket option is a European option on a portfolio of assets yielding a payoff which depends on the breaching behavior of the underlying basket of assets given an upper and a lower barrier with respect to the initial price of the portfolio of assets. The contract of a specific European double barrier basket option particularizes the payoff depending on whether the up- or down-barrier or neither of them is hit by the price of the underlying portfolio of assets - which can be the weighted sum or the average of the different assets. In particular, we will investigate European double barrier basket options on two underlying assets with an upper and a lower knock-out barrier featuring a finite number of cash settlements at predefined values of the underlyings between the strike and the upper barrier.

2.2 Types Of Traders

Participants of option markets have various aims and goals. Basically we group traders into three different classes: hedgers, speculators and arbitrageurs. We will give examples of how these types of traders (cf., e.g., [1, 5]) use options to achieve their purposes.

Hedgers: Many investors are risk-averse, i.e., they are unwilling to take large risks and use options as a measure to cover possible losses from future price changes.

Let us assume that there is an investor who owns shares of a certain company and fears losses due to a decline in the stock value. In this case, he could take a long position in put options (i.e., buy put options) with a strike price of the current stock price. He has to bear the costs of the put contract but this action guarantees that he can sell his shares in the future for at least the current stock price.

Speculators: Speculators try to use uncertain future price movements to gain profits by using options. They utilize their knowledge in the market to forecast future prices. Therefore, they bet on the asset price to go up or down. Furthermore, the leverage effect on options often makes it more attractive to hold an option instead of the share itself.

Let us assume an investor has 5000 US-D to invest and he picks a certain share whose price will presumably increase. He could either buy 100 shares worth 50 US-D or purchase 2500 call options with strike 55 US-D at the current option price 2 US-D. If the price of the stock rises as foreseen by the speculator, say up to 60 US-D, he will make a profit of $100 \cdot (60 - 50) = 1000$ US-D holding shares, but $2500 \cdot (60 - 55) - 2 \cdot 2500 = 7500$ US-D holding call options. On the other hand, if the strike price is not reached, he will lose the whole investment of 5000 US-D.

Arbitrageurs: Another group of participants in the option market are arbitrageurs who exploit price differences or wrong valuation of prices in markets to gain riskless profits by concurrently entering transactions in these markets.

Consider three financial assets in a financial market:

- a riskless bond with value $B_t = B(t)$ which is paid for at time $t = 0$ in months with $B_0 = 50$ US-D and results at maturity date $t = 1$ in months with interest rate $r = 0.1$ in $B_0(1 + r) = 55$ US-D,

- a stock with initial value 50 which attains one of the two possible states $S_T = 60$ US-D or $S_T = 40$ US-D at maturity date T,
- a call option with strike $K = 50$ US-D, maturity date T and option price $C_0 = 5$ US-D.

An arbitrageur would invest in a portfolio as follows: At time $t = 0$ he buys $2/5$ of the bond and one call option and sells (as short-selling is allowed) $1/2$ stock such that the value of his portfolio π is zero: $\pi_0 = 2/5 \cdot 50 + 1 \cdot 5 - 1/2 \cdot 50 = 0$. Since at maturity T the value of the stock can be either 40 US-D or 60 US-D, the value of the portfolio is either

$$\pi_T = 2/5 \cdot 60 + 1 \cdot 10 - 1/2 \cdot 60 = 4,$$

or

$$\pi_T = 2/5 \cdot 60 + 1 \cdot 0 - 1/2 \cdot 40 = 4.$$

Hence, the investor could realize an immediate riskless profit, because the price of the call option is too low. Options have to be appropriately priced so that arbitrage can be excluded. This leads to the put-call parity.

2.3 Put-Call Parity

More details about Put-Call Parity can be found in [1, 15]. Upper and lower bounds for the price of European call and put options can be derived under the assumption that the financial market is arbitrage-free, that the market is liquid, i.e., that there is a sufficiently large number of buyers and sellers such that changes in supply or demand only have little impact on the price, and frictionless, i.e., that there are neither transaction costs nor taxes, and that trade is possible at any time. In particular, denoting by T the maturity date, by K the strike, by $r > 0$ a fixed interest rate, and by S_t the spot price of the underlying asset, for the price $C(S_t;t)$ of a European put option we obtain

$$(S_t - K\ \exp(-r(T-t))_+ \leq C(S_t;t) \leq S_t, \quad 0 \leq t \leq T, \tag{2.3}$$

whereas for the the price $P(S_t;t)$ of a European put option there holds

$$(K\ \exp(-r(T-t) - S_t)_+ \leq P(S_t;t) \leq K\ \exp(-r(T-t)), \quad 0 \leq t \leq T. \tag{2.4}$$

Moreover, $C(S_t;t)$ and $P(S_t;t)$ are related by the so-called put-call parity

$$\pi(t) := (S_t + P(S_t;t) - C(S_t;t) = K\ \exp(-r(T-t)), \quad 0 \leq t \leq T. \tag{2.5}$$

The proofs of (2.3),(2.4), and (2.5) can be easily done by contradiction arguments. For instance, in order to verify (2.5), let us first assume that $\pi(t) < K\ \exp(-r(T-t))$. We buy the portfolio, i.e., we buy one share, one put option and sell one call option. Furthermore, we take a credit worth $K\ \exp(-r(T-t))$ and consequently save $K\ \exp(-r(T-t)) - \pi(t) > 0$. At maturity date, the value of the portfolio reads $\pi(t) = S_T + (K - S_T)_+ - (S_T - K)_+ = K$, which we bring to the bank to pay the credit. This leads to a risk-free profit of $K\ \exp(-r(T-t)) - \pi(t) > 0$ at time t contradicting the no-arbitrage principle. Likewise, if we assume $\pi(t) > K\ \exp(-r(T-t))$, then selling the portfolio, i.e., shorting one share, one put option, buying one call option and investing $K\ \exp(-r(T-t))$ in a riskless bond leads to a risk-free profit of $\pi(t) - K\ \exp(-r(T-t)) > 0$ contradicting the no-arbitrage principle as well, since at maturity T we get K from the bank and buy the portfolio at price $\pi(T) = K$.

2.4 Black-Scholes Equation

The derivation of the Black-Scholes equation(cf., e.g., [4, 9, 12, 32, 37, 39]) is based on the following assumptions on the financial market:

- there is no-arbitrage,
- the market is liquid and frictionless, i.e., there are no transaction costs and taxes,
- there are no dividends,
- the risk-free interest rate r to borrow and lend cash is constant in time, i.e., bonds $B_t, t \in \mathbb{R}_+$ satisfy $dB_t = rB_t dt$,
- it is possible to continuously buy any fraction of a security, i.e., bonds, shares, options, and short selling is permitted,
- the price of an asset satisfies the linear stochastic differential equation $dS_t = \mu S_t dt + \sigma S_t dW_t$ where $\mu \in \mathbb{R}$ is a constant drift parameter, $\sigma \in \mathbb{R}_+$ is the volatility of the asset and W_t is a Wiener process.

The idea, which is so-called Delta-Hedging, is to dynamically duplicate the option with a suitable portfolio which only consists of financial instruments whose values are known such as the value of the stock S and an investment or credit with interest rate r. In particular, a duplication portfolio is chosen such that this portfolio has the same value at maturity as the option. In this way, one can interpret the option price as a discounted expectation of the payoff at maturity T. It follows from the no-arbitrage principle and from the assumptions on the financial market that at each time the duplication portfolio has the same value as the option.

We consider a risk-free self-financing portfolio $R = R_t$ consisting of a bond $B = B_t$, a stock $S = S_t$, and a European option with value $y = y_t$. Changes in a self-financing portfolio are only financed by either buying or selling parts of the portfolio. From the above assumptions, Black and Scholes deduce the principle of risk-neutral valuation which implies that the present value of an option is the expected final value of the option discounted with the fixed interest rate r so that the drift parameter μ can be replaced by r which comes from the first assumption. With Itô's lemma, it can be shown that the value of the European option satisfies the following linear second order parabolic partial differential equation known as the Black-Scholes equation

$$\frac{\partial y}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 y}{\partial S^2} + rS\frac{\partial y}{\partial S} - ry = 0. \tag{2.6}$$

In order to guarantee a unique solution of (2.6), a final time condition and appropriate boundary conditions have to be taken into account depending on the type of the option. For European puts and calls, at maturity T the final condition is given according to

$$y(S,T) = \begin{cases} (S-K)_+ & \text{for a European call} \\ (K-S)_+ & \text{for a European put} \end{cases}. \tag{2.7}$$

The boundary conditions for $S = 0$ and $S \to \infty$ read as follows

$$y(0,t) = \begin{cases} 0 & \text{for a European call} \\ K\exp(-r(T-t)) & \text{for a European put} \end{cases}, \tag{2.8}$$

$$y(S,t) = O(S) \text{ for a European call}, \quad \lim_{S\to\infty} y(S,t) = 0 \text{ for a European put}.$$

The Black-Scholes equation (2.6) with final time condition (2.7) and boundary data (2.8) has the explicit solution

$$y(S,t) = S\phi(d_1) - K\ \exp(-r(T-t))\phi(d_2) \text{ for a European call,} \tag{2.9a}$$

and

$$y(S,t) = K\ \exp(-r(T-t))\phi(-d_2) - S\phi(-d_1) \text{ for a European call.} \tag{2.9b}$$

Here, ϕ denotes the cumulative distribution function of the standard normal distribution, and d_1, d_2 are given by

$$d_{1/2} = \frac{\ln(\frac{S}{K}) + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \tag{2.10}$$

Remark 2.1. *Extensions of the Black-Scholes model are able to take into account time-dependent interest rates* $r = r(t)$ *and temporally and spatially varying volatilities* $\sigma = \sigma(S_t, t)$ *as well as transaction costs and dividends.*

2.5 Multidimensional Black-Scholes Equation

Under the same assumptions on the financial market as in the previous subsection 2.4, we now consider a basket containing d assets whose prices $S_k = (S_{kt})_{t\geq 0}, 1 \leq k \leq d$, satisfy the following system of stochastic differential equations(cf., e.g., [26, 46])

$$dS_{kt} = S_{kt}\Big(\mu_k dt + \frac{\sigma_k}{\sqrt{1 + \sum\limits_{\ell \neq k} \rho_{k\ell}^2}}(dW_{kt} + \sum_{\ell=1}^{d} \rho_{k\ell} dW_{\ell t})\Big), \quad 1 \leq k \leq d. \tag{2.11}$$

Here, $\mu_k, 1 \leq k \leq d$, denotes the constant drift term of the k-th asset. The underlying Wiener process $(W_t)_{t\geq 0}$ is assumed to be multidimensional, $\sigma_k, 1 \leq k \leq d$, refers to the volatility of the k-th stock, and $\rho_{k\ell} \in [0,1], 1 \leq k, \ell \leq d$, stands for Pearson's correlation coefficient between stock k and ℓ. We assume that the correlation matrix

$$\xi := \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1d}\sigma_1\sigma_d \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2d}\sigma_2\sigma_d \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{d1}\sigma_d\sigma_1 & \rho_{d2}\sigma_d\sigma_2 & \cdots & \sigma_d^2 \end{pmatrix} \tag{2.12}$$

is symmetric and positive definite. Applying Itô's lemma to (2.11) as well as the principle of risk-neutral valuation leads to the solution

$$S_{kt} = S_{k0}\ \exp\Big((r - \frac{\sigma_k^2}{2})t + \frac{\sigma_k}{\sqrt{1 + \sum\limits_{\ell \neq k} \rho_{k\ell}^2}}(W_{kt} + \sum_{\ell \neq k} \rho_{k\ell} W_{\ell t})\Big), \quad 1 \leq k \leq d. \tag{2.13}$$

Under the assumptions on the financial market, for the price y of the basket option the following multidimensional Black-Scholes equation can be derived

$$\frac{\partial y}{\partial t} + \frac{1}{2} \sum_{k,\ell=1}^{d} \xi_{k\ell} S_k S_\ell \frac{\partial^2 y}{\partial S_k \partial S_\ell} + r \sum_{k=1}^{d} S_k \frac{\partial y}{\partial S_k} - ry = 0, \tag{2.14}$$

where $\xi_{k,\ell} := \rho_{k\ell}\sigma_k\sigma_\ell, 1 \leq k, \ell \leq d$. The equation (2.14) has to be complemented by a final time condition and boundary conditions depending on the type of option in much the same way as has been done in subsection 2.4.

2.6 Variational Formulation Of The Black-Scholes Equation

2.6.1 Weighted Sobolev Spaces

We use standard notation from Lebesgue and Sobolev space theory [42]. In particular, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, with boundary $\Gamma := \partial\Omega$, for $D \subseteq \Omega$ we refer to $L^p(D), 1 \leq p \leq \infty$ as the Banach spaces of p-th power integrable functions ($p < \infty$) and essentially bounded functions ($p = \infty$) on D with norm $\|\cdot\|_{L^p(D)}$. We denote by $L^p(D)_+$ the positive cone in $L^p(D)$, i.e., $L^p(D)_+ := \{v \in L^p(D) \mid v \geq 0 \text{ a.e. in } D\}$. In case $p = 2$, the space $L^2(D)$ is a Hilbert space whose inner product and norm will be referred to as $(\cdot,\cdot)_{L^2(D)}$.

For $m \in \mathbb{N}_0$ and weight functions $\omega = (\omega_\alpha)_{|\alpha|\leq m}$ with $\omega_\alpha \in L^\infty(D)_+, \alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d, |\alpha| := \sum_{i=1}^d \alpha_i$, we denote by $W_\omega^{m,p}(D)$ the weighted Sobolev spaces with norms

$$\|v\|_{W_\omega^{m,p}(D)} := \begin{cases} \Big(\sum\limits_{|\alpha|\leq m} \|\omega_\alpha D^\alpha v\|_{L^p(D)}^p\Big)^{1/p} & \text{, if } p < \infty \\ \max\limits_{|\alpha|\leq m} \|\omega_\alpha D^\alpha v\|_{L^\infty(D)} & \text{, if } p = \infty \end{cases},$$

and refer to $|\cdot|_{W_\omega^{m,p}(D)}$ as the associated seminorms. In particular, for $|\alpha| = 1$ we use the notation $\nabla_\omega v := (S_1 \partial v/\partial S_1, \cdots, S_d \partial v/\partial S_d)^T$. For $p < \infty$ and $s \in \mathbb{R}_+, s = m + \sigma, m \in \mathbb{N}_0, 0 < \sigma < 1$, we define the weighted Sobolev space $W_\omega^{s,p}(D)$ with norm $\|\cdot\|_{W_\omega^{s,p}(D)}$ in analogy to the standard, non-weighted case and refer to $W_{\omega,0}^{s,p}(D)$ as the closure of $C_0^\infty(D)$ in $W_\omega^{s,p}(D)$. For $s < 0$, we denote by $W_\omega^{-s,p}(D)$ the dual space of $W_{\omega,0}^{-s,q}(D), p^{-1} + q^{-1} = 1$. In case $p = 2$, the spaces $W_\omega^{s,2}(D)$ are Hilbert spaces. We will write $H_\omega^s(D)$ instead of $W_\omega^{s,2}(D)$ and refer to $(\cdot,\cdot)_{H_\omega^s(D)}$ and $\|\cdot\|_{H_\omega^s(D)}$ as the inner products and associated norms. In the standard case $\omega_\alpha \equiv 1, |\alpha| \leq m$, we will drop the subindex ω.

For a Banach space X and its dual X^*, we refer to $\langle\cdot,\cdot\rangle_{X^*,X}$ as the dual pairing between

X^* and X. For Banach spaces $X_i, 1 \leq i \leq n, n \in \mathbb{N}$, and a function $v \in \bigcap_{i=1}^n X_i$, we refer to $\|v\|_{\bigcap_{i=1}^n X_i}$ as the norm

$$\|v\|_{\bigcap_{i=1}^n X_i} := \max_{i \leq i \leq n} \|v\|_{X_i}. \tag{2.15}$$

Moreover, for $T > 0$ and a Banach space X, we denote by $L^p((0,T),X), 1 \leq p \leq \infty$, and $C([0,T],X)$ the Banach spaces of functions $v : [0,T] \to X$ with norms

$$\|v\|_{L^p((0,T),X)} := \begin{cases} \left(\int\limits_0^T \|v(t)\|_X^p dt \right)^{1/p} , \ 1 \leq p < \infty \\ \text{ess sup}_{t \in [0,T]} \|v(t)\|_X \ , \ p = \infty \end{cases} , \quad \|v\|_{C([0,T],X)} := \max_{t \in [0,T]} \|v(t)\|_X.$$

The spaces $W^{s,p}((0,T),X)$ and $H^s((0,T),X), s \in \mathbb{R}_+$, are defined likewise.
In particular, for a subspace $V \subset H^1_\omega(\Omega)$ with dual V^* we will consider the space

$$H^1((0,T),V^*) \cap L^2((0,T),V), \tag{2.16}$$

and note that the following continuous embedding holds true

$$H^1((0,T),V^*) \cap L^2((0,T),V) \subset C([0,T],L^2(\Omega)). \tag{2.17}$$

For $y \in H^1((0,T),V^*) \cap L^2((0,T),V)$, we further denote by $\gamma_{\Sigma'}(y), \Sigma' \subset \Sigma := \Gamma \times (0,T)$, the trace of y on Σ'.

2.6.2 Weak Solution Of The Black-Scholes Equation

We consider a European Basket Put Option $y = y(S,t)$ with strike K and maturity date $T > 0$ on two underlying assets $S = (S_1, S_2)^T$ in $Q := \Omega \times (0,T)$. The spatial domain Ω

is supposed to be the rectangle $\Omega := (0, S_1^{max}) \times (0, S_2^{max})$ with boundary $\Gamma := \sum_{\nu=1}^{4} \bar{\Gamma}_\nu$, where $\Gamma_1 := (0, S_1^{max} \times \{0\}, \Gamma_2 := \{0\} \times (0, S_2^{max}), \Gamma_3 := \{S_1^{max}\} \times (0, S_2^{max})$, and $\Gamma_4 := (0, S_1^{max}) \times \{S_2^{max}\}$. We assume that S_1^{max} and S_2^{max} are chosen sufficiently large such that $y(\cdot,t)|_{\Gamma_\nu} = 0, t \in (0,T), 3 \le \nu \le 4$. The price of the option satisfies the following final time/boundary value problem for the two-dimensional Black-Scholes equation

$$\frac{\partial y}{\partial t} + \frac{1}{2} \sum_{k,\ell=1}^{2} \xi_{k\ell} S_k S_\ell \frac{\partial^2 y}{\partial S_k \partial S_\ell} + r \sum_{k=1}^{2} S_k \frac{\partial y}{\partial S_k} - ry = 0 \quad \text{in } Q, \tag{2.18a}$$

$$y = g_\nu \quad \text{on } \Sigma_\nu := \Gamma_\nu \times (0,T),\ 1 \le \nu \le 4, \tag{2.18b}$$

$$y(\cdot,T) = y^T \quad \text{in } \Omega, \tag{2.18c}$$

where $\xi = (\xi_{k\ell})_{k,\ell=1}^2$ is the correlation matrix from (2.12) which now may depend on S and t, $r = r(t)$ stands for the interest rate and the boundary data $g_\nu, 1 \le \nu \le 4$, as well as the the final time data y^T are given by

$$g_1(\cdot,t) := (K \exp(-r(T-t)) - S_1)_+, \quad g_2(\cdot,t) := (K \exp(-r(T-t)) - S_2)_+,$$
$$g_3(\cdot,t) = g_4(\cdot,t) := 0, \quad y^T := (K - (S_1 + S_2))_+.$$

For the weak formulation of (2.18a)-(2.18c) we assume $\xi_{k\ell} \in L^\infty((0,T); W^{1,\infty}(\Omega)), 1 \le k,\ell \le 2$, and the existence of a constant $\xi_{min} > 0$ such that for all $\eta \in \mathbb{R}^2$ there holds

$$\sum_{k,\ell=1}^{2} \xi_{k\ell}(S,t)\eta_k\eta_\ell \ge \xi_{min}\ |\eta|^2 \quad \text{f.a.a. } (S,t) \in Q.$$

Moreover, we suppose that $r \in L^\infty(0,T)$ with $r(t) > 0$ for almost all $t \in (0,T)$.

Setting

$$W(0,T) := \{w \in H^1((0,T); H^1_\omega(\Omega)^*) \cap L^2((0,T); H^1_\omega(\Omega)) \mid \gamma_{\Sigma_\nu}(y) = g_\nu, 1 \le \nu \le 4\},$$

the variational or weak formulation of (2.18a)-(2.18c) amounts to the computation of $y \in W(0,T)$ such that for all $v \in L^2((0,T); H^1_{\omega,0}(\Omega))$ there holds

$$\int_0^T \langle \frac{\partial y}{\partial t}, v\rangle_{H^{-1}_\omega(\Omega), H^1_{\omega,0}(\Omega)} \; dt - \int_0^T a(t; y, v) \; dt = 0, \tag{2.19a}$$

$$y(\cdot, T) = y^T. \tag{2.19b}$$

Here, the bilinear form $a(t;\cdot,\cdot), t \in (0,T)$, is given by

$$a(t;y,v) := \int_\Omega \Big(\frac{1}{2}\sum_{k,\ell=1}^{2} \xi_{k\ell} S_k \frac{\partial y}{\partial S_k} S_\ell \frac{\partial v}{\partial S_\ell} - \sum_{k=1}^{2} r S_k \frac{\partial y}{\partial S_k} v - \Big(\frac{1}{2}\sum_{k,\ell=1}^{2} (S_k S_\ell \frac{\partial \xi_{k\ell}}{\partial S_\ell} + \xi_{k\ell} S_k) - r\Big) y v\Big) dS.$$

Theorem 2.2. *Under the above assumptions on the data ξ, r, and $g_\nu, 1 \le \nu \le 4$, as well as y^T,the variational problem* (2.19a),(2.19b) *admits a unique solution $y \in W(0,T) \subset C([0,T]; L^2(\Omega)$ which continuously depends on the data.*

Proof. The bilinear form $a(t;\cdot,\cdot)$ satisfies the Gårding inequality

$$a(t;v,v) \le \alpha \; \|v\|^2_{H^1_\omega(\Omega)} - \beta \; \|v\|^2_{L^2(\Omega)} \quad \text{f.a.a. } t \in (0,T),$$

for some $\alpha > 0$ and $\beta \ge 0$. Then, the existence of a solution can be shown using the Galerkin method, i.e., by considering a semi-discretization in space by means of a suitably chosen family of dense subspaces. The uniqueness follows by standard arguments, and the continuous dependence on the data results from a Gronwall-type inequality. For details we refer to [16],[37],[43], or . $\square$

Chapter 3

Hedging With Options And Futures Contracts

In this chapter, we will be concerned with hedging strategies. After a brief introduction to Greeks and futures contracts in sections 3.1 and 3.2, we will first consider Delta hedging with options and futures contracts (sections 3.3 and 3.4) as standard hedging tools and then concentrate on Delta hedging with European Double Barrier Options as an attractive alternative which combines the advantages of hedging with options and futures contracts (section 3.5).

3.1 Greeks

A matter of particular interest for hedging portfolios are sensitivities of the option price that describe changes in the value y, if there is a change in one of the underlying parameters and variables while the other parameters and variables remain constant. In risk management, these hedge sensitivities are called Greeks(cf., e.g., [15, 23]). We briefly recall the most important Greeks:

Delta: the Delta $\Delta = \partial y/\partial S$ indicates the rate of change of the option price with respect to the price of the underlying asset.

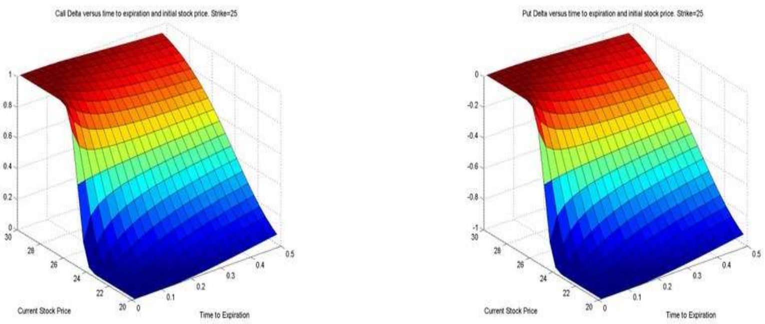

Figure 3.1: European Vanilla Call Delta (left) and Put Delta (right) as a function of the time to expiration and the initial price of the underlying (K=25).

Gamma: the Gamma $\Gamma = \partial^2 y/\partial S^2$ is the sensitivity of the Delta with respect to the underlying asset.

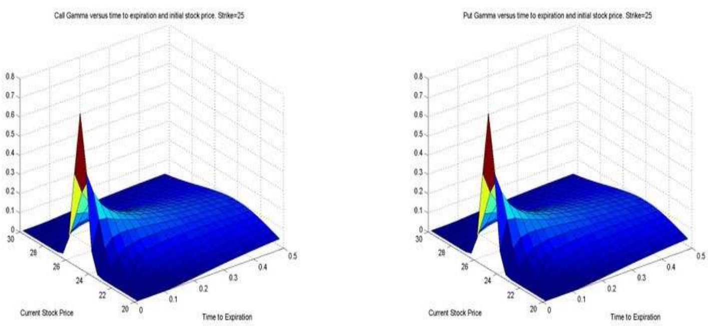

Figure 3.2: European Vanilla Call Gamma (left) and Put Gamma (right) as a function of the time to expiration and the initial price of the underlying (K=25).

Rho: the Rho $\rho = \partial y / \partial r$ is referred to as the rate of change of the option's value with respect to the interest rate.

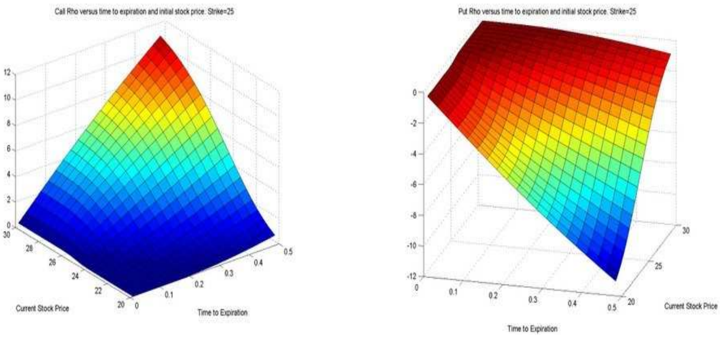

Figure 3.3: European Vanilla Call Rho (left) and Put Rho (right) as a function of the time to expiration and the initial price of the underlying (K=25).

Theta: the Theta $\Theta = \partial y/\partial t$ is the time decay of an option, i.e., the rate of change of the value of the option price with abbreviated maturity.

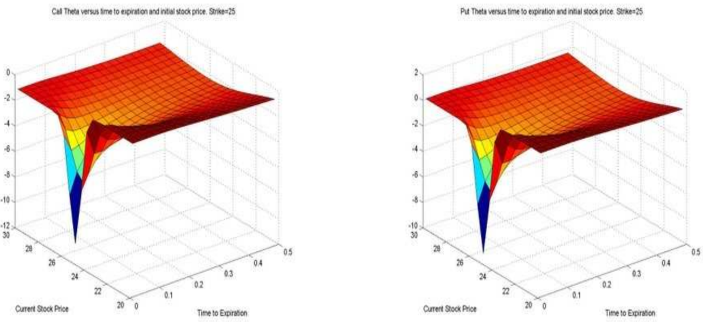

Figure 3.4: European Vanilla Call Theta (left) and Put Theta (right) as a function of the time to expiration and the initial price of the underlying (K=25).

Vega: the Vega $\kappa = \partial y/\partial \sigma$ measures the sensitivity with respect to the volatility.

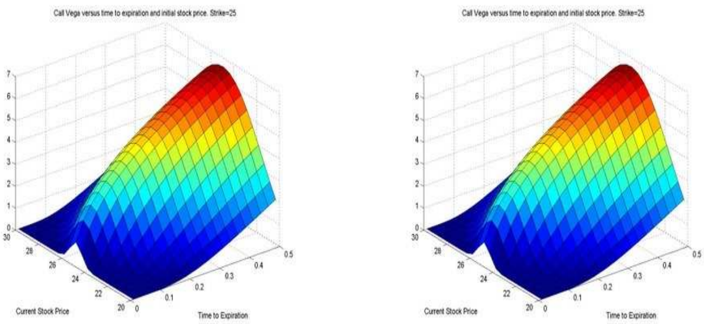

Figure 3.5: European Vanilla Call Vega (left) and Put Vega (right) as a function of the time to expiration and the initial price of the underlying (K=25).

Note that the Call Vega and the Put Vega are always the same.

3.2 Futures Contracts

A futures contract is a contract between two parties to buy or sell a certain amount of an asset (e.g., commodities, currencies, securities, or stock indices) at a fixed date in the future at a prespecified price. The contracts are traded at a futures exchange such as the CME group (formerly Chicago Mercantile Exchange). The party which agrees to buy the assets in the future assumes a long position, whereas the other party assumes a short position. The future date is referred to as the delivery date or fixed settlement date, and the official price of the futures contract at the end of the day's trading session is called the settlement price. As opposed to options, in case of a futures contract the holder of the contract has the obligation to deliver or receive the assets, i.e., both parties of the contract must fulfill the contract on the settlement day. The assets are provided either physically (physical settlement) or in cash (cash settlement). In order to minimize counterparty risk to traders, trades on regulated futures exchanges are guaranteed by a clearing house which becomes the buyer to each seller and the seller to each buyer. Moreover, in order to minimize credit risk to the exchange, traders are assumed to post a margin which typically amounts to 5 % - 15 % of the value of the futures contract. The margin consists of an initial margin, established by the futures exchange on the maximum estimated change in contract value within a trading day, and a variation or maintenance margin, established by the broker to restore the amount of the available initial margin due to changes in the market price of the asset and in the contract value. The variation margin is computed on a daily basis and calls for that margin by the broker are expected to be paid and received on the same day. Otherwise, the broker may close sufficient positions to satisfy the amount of the margin call.

3.3 Delta Hedging With Options

We want to illustrate the principle of delta hedging, i e., eliminating the risk for the writer of an option by purchasing the underlying asset: Say, a reinsurance company is the writer of 1000 call options worth $C = 5$ US-D per stock. Assuming that $\Delta = 0.5$, that the value of the underlying asset increases by one point $\delta S = 1.0$, and that the Delta remains constant during this tiny interval, the option writer loses $1000 \cdot \Delta \cdot \delta S = 500$ points. Delta hedging avoids this loss. The reinsurance company hedges by purchasing $\Delta \cdot 1000 = 500$ stocks. Then, its stock portfolio gains 500 points and loses 500 from the option contracts, i.e., the value of the whole portfolio remains constant. However, the Delta of an option is not constant due to fluctuations in the stock price and time to maturity. Therefore, the portfolio has to be re-balanced perpetually. These adaptations of the portfolio can be expensive and tedious.

3.4 Delta Hedging With Futures Contracts

We illustrate hedging with Dow Jones Industrial Average Futures by the following scenario:

Scenario: A US-based insurance company has a Dow Jones Industrial Average (DJIA)-like stock portfolio worth 10^9 US-D. The DJIA is 12000 points. The company predicts decreasing stock values and wants to safeguard the portfolio, i.e., to achieve a risk-free portfolio. Regulations or market conditions prevent the company from selling stocks.

Delta Hedging: The insurance company decides to open a Dow Jones Industrial Average Futures (FDJIA) short position, i.e., to sell FDJIA at the CME (Chicago Mercantile

Exchange). Assume that one such short position has a profit/loss $\Delta_{FDJIA} = 0.5$. Consequently, the insurance company has to sell

$$\frac{10^9}{0.5 \cdot 12000} \approx 166667 \text{ FDJIA.}$$

The clearing agency immediately demands an initial margin per FDJIA-long as well as a safety margin in addition to that. Moreover, in the event of adverse price movements, the selling party has to deposit more cash or securities into its margin account at the exchange. If the company is unable to make the necessary deposit, the company is impelled to close out its position prematurely. Moreover, the permanent margin adaption at the clearing agency is quite troublesome. Another disadvantage is the margin variation depending on the money volatility of options on the DJIA which may oscillate considerably. As reference, figures 3.6, 3.7, and 3.8 which comes from the historical data on the website display the implied three-months (six-months, one-year) at the money volatility of options on the DJIA issued by the Chicago Board Options Exchange (CBOE).

Figure 3.6: Implied three-months at the money volatility (CBOE) of options on the DJIA

Figure 3.7: Implied six-months at the money volatility (CBOE) of options on the DJIA

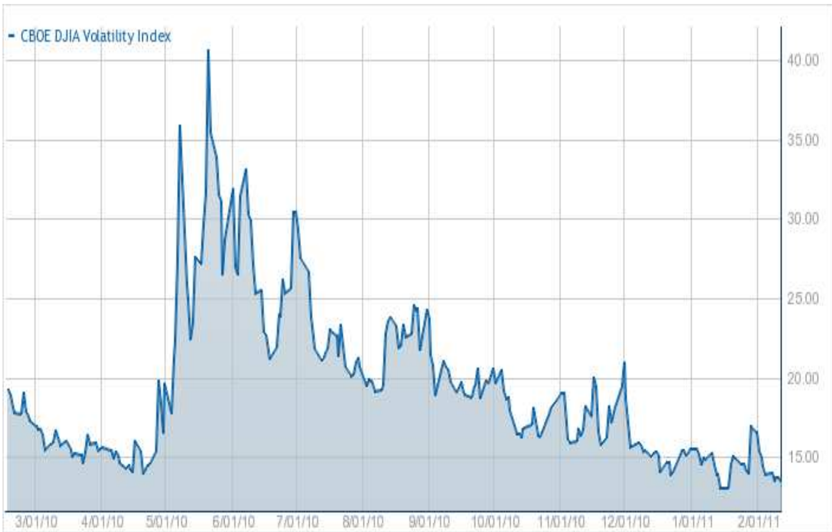

Figure 3.8: Implied one-year at the money volatility (CBOE) of options on the DJIA

3.5 Hedging With European Double Barrier Options

We refer to [9, 32, 8, 13] for more details about this section. European double barrier options with optimized cash settlements are able to combine the advantages of both futures and options. We consider a DJIA European call option $C = C(S,t)$ for $(S,t) \in Q := (DJIA_{min}, DJIA_{max}) \times (0,T)$ which satisfies the Black-Scholes equation (2.6). For the profit/loss Δ per DJIA point there holds $\Delta = \partial C / \partial S$. Assuming a constant Delta $\Delta = \Delta_{opt} > 0$, it follows that $\partial^2 C / \partial S^2 = 0$ such that the Black-Scholes equation simplifies to the first order PDE

$$\frac{\partial C_{\Delta_{opt}}}{\partial t} + rS \, \frac{\partial C_{\Delta_{opt}}}{\partial S} - rC_{\Delta_{opt}} = 0 \quad \text{in } Q, \tag{3.1a}$$

with the boundary conditions

$$C_{\Delta_{opt}}(DJIA_{min}, t) = \Delta_{opt} \; DJIA_{min} \; \Big(1 - \exp(-r(T-t))\Big), \quad t \in (0,T), \tag{3.1b}$$

$$C_{\Delta_{opt}}(DJIA_{max}, t) = \Delta_{opt} \; \Big(DJIA_{max} - DJIA_{min} \; \exp(-r(T-t))\Big), \quad t \in (0,T), \tag{3.1c}$$

and the final time condition

$$C_{\Delta_{opt}}(S, T) = \Delta_{opt} \; DJIA_{min} \; \Big(1 - \exp(-r(T-t))\Big), \quad t \in (0,T), \tag{3.1d}$$

$$C_{\Delta_{opt}}(DJIA_{max}, t) = \Delta_{opt} \; \Big(S - DJIA_{min}\Big), \quad S \in (DJIA_{min}, DJIA_{max}). \tag{3.1e}$$

The analytical solution of (3.1a)-(3.1e) is given by

$$C_{\Delta_{opt}}(S, t) = \Delta_{opt} \; \Big(S - DJIA_{min} \; \exp(-r(T-t))\Big), \quad (S,t) \in Q. \tag{3.2}$$

We note that in case of a knock-out European Double Barrier Call with the lower barrier $DJIA_{min}$ and the upper barrier $DJIA_{max}$ the boundary conditions (3.1d),(3.1c) and the final time condition (3.1e) correspond to the cash settlements at the option's expiration, i.e., when one of the barriers is hit or the maturity date T is reached.

Recalling the example from subsections 3.3 and 3.4, the insurance company sells 333333 DJIA European Double Barrier calls and accepts the obligation to pay the cash settlements (3.1d),(3.1d), or (3.1e) at the expiration date. The company's premium per call is 768.24 US-D for a strike $DJIA_{min} = 11000$, if $S = 12000$, $r = 0.05$, and $T = 1$ year, which is the only payment during the option's lifetime. The buyer of 100 options synthesizes 1 DJIA-long position costing 76824 US-D, whereas the insurance company synthesizes 1 DJIA-short position.

As mentioned earlier, a constant Delta is not realistic. In the following chapter, we will consider how to choose the cash settlements such that the Delta comes as close to a constant as possible.

Chapter 4

Optimal Control Of European Double Barrier Basket Options

The exposition of this chapter follows closely that of [22]. In this chapter, we will discuss hedging of European Double Barrier Basket Options with two assets featuring multiple cash settlements at the option's expiry date and formulate the hedging as an optimal control problem for the two-dimensional Black-Scholes equation with a tracking type objective functional and the cash settlements as controls. In particular, we will derive the optimality conditions in terms of the state, the control, and the adjoint state (section 4.2, cf., e.g., [19, 25, 27, 45]). For numerical purposes, we consider a discretization of the optimal control problem using P1 conforming finite elements with respect to a simplicial triangulation of the spatial domain and the implicit Euler scheme for discretization in time with respect to a partition of the time interval(cf.,e.g., [35]).

4.1 Hedging With European Double Barrier Basket Options

We consider a European Double Barrier Basket Call Option on a basket of two assets with prices S_1 and S_2, maturity date $T > 0$, strike $K > 0$, and barriers K_{min}, K_{max} such that $K_{min} < K < K_{max}$. The spatial domain $\Omega \subset \mathbb{R}^2_+$ for the price $y(S,t), S = (S_1, S_2) \in \Omega, t \in [0,T]$, of the option is the trapezoidal domain (cf. Figure 3.1) $\Omega := \{S = (S_1, S_2) \in \mathbb{R}^2_+ \mid K_{min} < |S| := S_1 + S_2 < K_{max}\}$ with boundaries $\Gamma_1 := (K_{min}, K_{max}) \times \{0\}, \Gamma_2 := \{0\} \times (K_{min}, K_{max}), \Gamma_3 := \{S \in \mathbb{R}^2_+ \mid |S| = K_{min}\}$, and $\Gamma_4 := \{S \in \mathbb{R}^2_+ \mid |S| = K_{max}\}$ (cf. Figure 4.1).

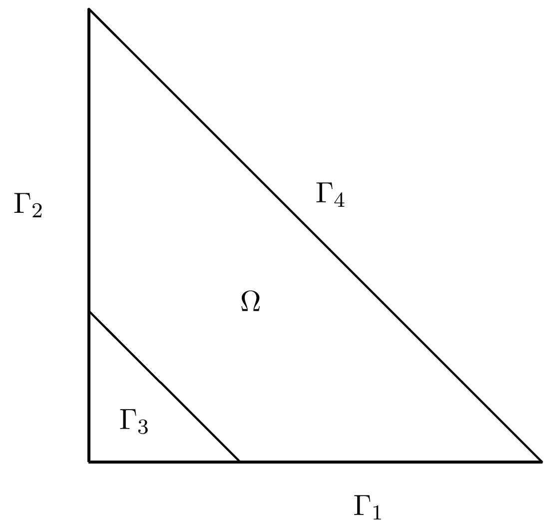

Figure 4.1: Spatial domain for European Double Barrier Basket Option.

We refer to $r = r(t), t \in [0,T]$, as the risk-free interest rate and to $\sigma_k = \sigma_k(S,t), 1 \leq k \leq 2, S \in \Omega, t \in [0,T]$, as the volatilities of the assets. Moreover, $\rho = (\rho_{k\ell})^2_{k,\ell=1}$ with $\rho_{kk} = 1, 1 \leq k \leq 2$, and $\rho_{12} = \rho_{21} = 2\theta/(1+\theta^2), -1 < \theta < +1$, are the correlations and $\xi = (\xi_{k\ell})^2_{k,\ell=1}, \xi_{k\ell} := \rho_{k\ell}\sigma_k\sigma_\ell, 1 \leq k, \ell \leq 2$, is the correlation matrix. We impose the following regularity assumptions on the data:

Assumption 1: $\sigma_k \in C([0,T], C^2(\overline{\Omega})), 1 \leq k \leq 2$, and there exist constants $\sigma_k^{(min)} > 0, C_{\sigma_k} > 0$, such that

$$\sigma_k(S,t) \geq \sigma_k^{(min)} \quad , \quad (S,t) \in \bar{Q}, 1 \leq k \leq 2, \tag{4.1a}$$

$$|S \cdot \nabla \sigma_k(S,t)| \leq C_{\sigma_k} \quad , \quad (S,t) \in \bar{Q}, 1 \leq k \leq 2. \tag{4.1b}$$

Assumption 2: $r \in C([0,T])$ such that $r(t) > 0, t \in [0,T]$.

Remark 4.1. *It is an immediate consequence of* **Assumption 1** *that the correlation matrix satisfies* $\xi_{k,\ell} \in C([0,T], C^2(\overline{\Omega})), 1 \leq k, \ell \leq 2$, *and that there exists a constant* $\xi_{min} > 0$ *such that for all* $\eta \in \mathbb{R}^2$ *there holds*

$$\sum_{k,\ell=1}^{2} \xi_{k,\ell}(S,t)\eta_k\eta_\ell \geq \xi_{min}|\eta|^2 \quad , \quad (S,t) \in \bar{Q}. \tag{4.2}$$

It is well-known [22, 44] that the price $y = y(S,t), (S,t) \in Q := \Omega \times (0,T)$, of the option satisfies the following final time/boundary value problem for the two-dimensional Black-Scholes equation:

$$\frac{\partial y}{\partial t} + A(t)y = 0 \quad \text{in } Q := \Omega \times (0,T), \tag{4.3a}$$

$$y = y_\nu \quad \text{on } \Sigma_\nu := \Gamma_\nu \times (0,T) \ , \ 1 \leq \nu \leq 4, \tag{4.3b}$$

$$y(\cdot, T) = y^T \quad \text{in } \Omega. \tag{4.3c}$$

Here, $A(t), t \in [0,T]$, refers to the time-dependent second order elliptic operator

$$A(t) := \frac{1}{2} \sum_{k,\ell=1}^{2} \xi_{k\ell} S_k S_\ell \frac{\partial^2}{\partial S_k \partial S_\ell} + r \sum_{k=1}^{2} S_k \frac{\partial}{\partial S_k} - r. \tag{4.4}$$

The final time data y^T at maturity date T is given by the payoff

$$y^T(S) := (\|S\| - K)_+ \ , \ S \in \Omega. \tag{4.5}$$

Further, $y_3 = 0$ which means that the option expires worthlessly at the lower bound, and the constant y_4 represents a cash settlement at the upper barrier Σ_4. The boundary values $y_\nu, 1 \leq \nu \leq 2$, are the solutions of the one-dimensional Black-Scholes equations

$$\frac{\partial y_\nu}{\partial t} + A_\nu(t) y_\nu = 0 \quad \text{in } \Sigma_\nu := \Gamma_\nu \times (0,T), \tag{4.6a}$$

$$y_\nu(S_\nu, t) = \begin{cases} 0 \ , \ S_\nu = K_{min} \\ y_4 \ , \ S_\nu = K_{max} \end{cases} \ , \ t \in (0,T), \tag{4.6b}$$

$$y_\nu(\cdot, T) = y^T|_{\Gamma_\nu} \quad \text{in } \Gamma_\nu. \tag{4.6c}$$

Here, $A_\nu(t), 1 \leq \nu \leq 2, t \in [0,T]$, are the time-dependent second order elliptic operators

$$A_\nu(t) := \frac{1}{2}\sigma_\nu^2 S_\nu^2 \frac{\partial^2}{\partial S_\nu^2} + r S_\nu \frac{\partial}{\partial S_\nu} - r. \tag{4.7}$$

As a particular feature, we consider additional cash settlements at instances between the strike K and the upper bound K_{max} (cf. Figure 4.2). To this end, we provide a partition $K =: K_0 < K_1 < \cdots < K_M := K_{max}, M \in \mathbb{N}$, of the interval $[K, K_{max}]$, where $K_i := K + i\delta_{|S|}, 0 \leq i \leq M, \ \delta_{|S|} := (K_{max} - K)/M$. We set $\mathbf{u} := (u_1, \cdots, u_M)^T \in \mathbb{R}_+^M$, and define

$$\begin{aligned} &(g(\mathbf{u}))(S) = u_{i-1} g_1^{(i)}(S) + u_i g_2^{(i)}(S) \text{ for } |S| \in [K_{i-1}, K_i], \ i = 0, \cdots, M, \\ &g_1^{(i)}(S) := (K_i - S)/\delta_{|S|} \ , \ g_2^{(i)}(S) := (S - K_{i-1})/\delta_{|S|}, \end{aligned} \tag{4.8}$$

where for notational convenience we have set $K_{-1} = K_{min}, u_{-1} = u_0 = 0$. On this basis, we choose $y_4 = u_M$ and $y^T = g(\mathbf{u})$.

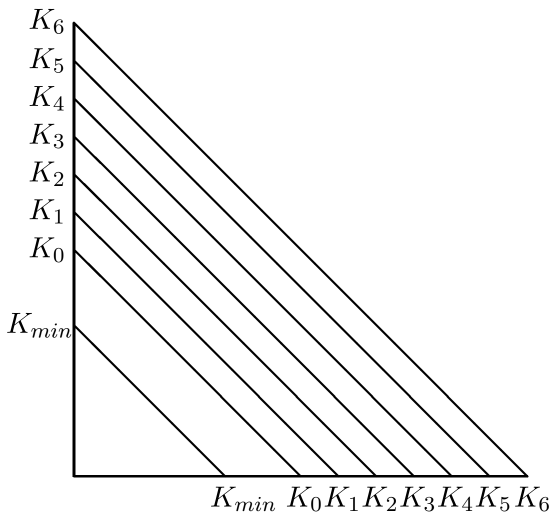

Figure 4.2: Cash settlements with respect to $K_i, 1 \leq i \leq M, M = 6$.

We consider the cash settlements $\mathbf{u}$ in (4.8) as a control vector that has to be chosen such that the Greek $\mathbf{\Delta} := \nabla y$ per asset point is as close to a prespecified profit $d = (d_1, d_2)^T$ as possible. For given bounds $0 < \alpha_i < \beta_i, 1 \leq i \leq M$, the controls are subject to the constraints

$$\mathbf{u} \in \mathbf{U}_{ad} := \{\mathbf{v} = (v_1, \cdots, v_M)^T \in \mathbb{R}_+^M \mid \alpha_i \leq v_i \leq \beta_i,\ 1 \leq i \leq M\}. \tag{4.9}$$

Consequently, the hedging with European Double Barrier Basket Options featuring multiple cash settlements $\mathbf{u}$ can be stated as the following optimal control problem for the two-dimensional Black-Scholes equation:

Find $(y, \mathbf{u})$ such that

$$\inf_{y,\mathbf{u}} J(y_Q, u) := \frac{1}{2} \int_0^T \int_\Omega |\nabla y - d|^2 dS dt, \tag{4.10}$$

subject to (4.3a)-(4.3c),(4.6a)-(4.6c), and (4.9).

For the variational formulation of the optimal control problem, we first reformulate the final time/boundary value problems for the backward parabolic equations as initial/boundary value problems:

$$\frac{\partial y}{\partial t} - A(t)y = 0 \quad \text{in } Q := \Omega \times (0,T), \tag{4.11a}$$

$$y = \begin{cases} y_\nu \ , \text{ on } \Sigma_\nu := \Gamma_\nu \times (0,T) \ , \ 1 \le \nu \le 2, \\ 0 \ , \text{ on } \Sigma_3 := \Gamma_3 \times (0,T) \\ u_M \ , \text{ on } \Sigma_4 := \Gamma_4 \times (0,T) \end{cases} , \tag{4.11b}$$

$$y(\cdot,0) = g(\mathbf{u}) \quad \text{in } \Omega, \tag{4.11c}$$

$$\frac{\partial y_\nu}{\partial t} - A_\nu(t)y_\nu = 0 \quad \text{in } \Sigma_\nu, \tag{4.12a}$$

$$y_\nu(S_\nu,t) = \begin{cases} 0 \ , \ S_\nu = K_{min} \\ u_M \ , \ S_\nu = K_{max} \end{cases} , \ t \in (0,T), \tag{4.12b}$$

$$y_\nu(\cdot,0) = g(\mathbf{u})|_{\Gamma_\nu} \quad \text{in } \Gamma_\nu. \tag{4.12c}$$

We note that for notational simplicity we have kept the same notation for y and y_ν as well as for the operators $A(t), A_\nu(t), 1 \le \nu \le 2$.

For the weak formulations of the initial/boundary value problems (4.11a)-(4.11c) and

(4.12a)-(4.12c) we introduce the function spaces

$$\begin{aligned}
W(0,T) &:= H^1((0,T),V^*) \cap L^2((0,T),V),\\
V &:= \{v \in H^1_\omega(\Omega) \mid v|_{\Sigma_{nu}} = y_\nu, 1 \le \nu \le 2, v|_{\Sigma_3} = 0, v_{\Sigma_4} = u_M\},\\
W_\nu(0,T) &:= H^1((0,T),V_\nu^*) \cap L^2((0,T),V_\nu), \quad 1 \le \nu \le 2,\\
V_\nu &:= \{v \in H^1_\omega(\Sigma_\nu) \mid v(K_{min}) = 0, v(K_{max}) = u_M\}, \quad 1 \le \nu \le 2,
\end{aligned}$$

as well as the bilinear forms $a(t;\cdot,\cdot), t \in (0,T)$, and $a_\nu(t;\cdot,\cdot), t \in (0,T), 1 \le \nu \le 2$, according to

$$\begin{aligned}
a(t;y,v) := \int\limits_\Omega \Big(&\frac{1}{2}\sum_{k,\ell=1}^{2} \xi_{k\ell} S_k \frac{\partial y}{\partial S_k} S_\ell \frac{\partial v}{\partial S_\ell} - \\
&\sum_{k=1}^{2} r S_k \frac{\partial y}{\partial S_k} v - \Big(\frac{1}{2}\sum_{k,\ell=1}^{2} (S_k S_\ell \frac{\partial \xi_{k\ell}}{\partial S_\ell} + \xi_{k\ell} S_k) - r\Big) y v\Big) dS,
\end{aligned}$$

and

$$\begin{aligned}
a_\nu(t;y_\nu,v_\nu) := \int\limits_{\Gamma_\nu} \Big(&\frac{1}{2}\sigma_\nu^2 S_\nu \frac{\partial y_\nu}{\partial S_\nu} S_\nu \frac{\partial v_\nu}{\partial S_\nu} - \\
&r S_\nu \frac{\partial y_\nu}{\partial S_\nu} v_\nu - (\sigma_\nu^2 S_\nu + \sigma_\nu S_\nu^2 \frac{\partial \sigma_\nu}{\partial S_\nu} - r) y_\nu v_\nu\Big) dS_\nu.
\end{aligned}$$

A function $y \in W(0,T)$ is called a weak solution of (4.11a)-(4.11c), if for all $v \in L^2((0,T), H^1_{\omega,0}(\Omega))$ there holds

$$\int\limits_0^T \langle \frac{\partial y}{\partial t}, v\rangle_{H^{-1}_\omega(\Omega),H^1_{\omega,0}(\Omega)} dt + \int\limits_0^T a(t;y,v) dt = 0, \tag{4.13a}$$

$$y(\cdot,0) = g(\mathbf{u}) \tag{4.13b}$$

Likewise, a function $y_\nu \in W_\nu(0,T)$ is said to be a weak solution of (4.12a)-(4.12c), if for all $v_\nu \in L^2((0,T), H^1_{\omega,0}(\Gamma_\nu))$ there holds

$$\int_0^T \langle \frac{\partial y_\nu}{\partial t}, v_\nu\rangle dt + \int_0^T a_\nu(t; y_\nu, v_\nu)dt = 0, \tag{4.14a}$$

$$y_\nu(\cdot, 0) = g(\mathbf{u})|_{\Gamma_\nu}. \tag{4.14b}$$

The existence and uniqueness of weak solutions and their regularity properties can be deduced as, for instance, in [1]. In particular, we have the following result:

Theorem 4.2. *For any admissible control* $\mathbf{u} \in \mathbf{U}_{ad}$*, the state equations* (4.13a),(4.13b) *and* (4.14a),(4.14b) *admit solutions satisfying*

$$y \in C([0,T], V) \cap L^2((0,T), V \cap H^2_\omega(\Omega)), \tag{4.15a}$$

$$y_\nu \in C([0,T], V_\nu) \cap L^2((0,T), V_\nu \cap H^2_\omega(\Gamma_\nu)), 1 \le \nu \le 2. \tag{4.15b}$$

Moreover, the solutions depend continuously on the data in the following sense:

$$exp(-2\lambda t)\|y(t)\|^2_{L^2(\Omega)} + 2\xi^2_{min} \int_0^t exp(-2\lambda\tau)|y(\tau)|^2_V \, d\tau \le \|g(\mathbf{u})\|^2_{L^2(\Omega)}, \tag{4.16}$$

$$exp(-2\lambda_\nu t)\|y_\nu(t)\|^2_{L^2(\Gamma_\nu)} + \frac{1}{2}\, (\sigma_\nu^{(min)})^2 \int_0^t exp(-2\lambda_\nu\tau)|y_\nu(\tau)|^2_{V_\nu} \, d\tau \le \|g(\mathbf{u})\|^2_{L^2(\Gamma_\nu)}.$$

Proof. Observing assumptions 1,2 and taking the Poincaré-Friedrichs inequalities for weighted Sobolev spaces into account, we deduce that the bilinear forms $a(t;\cdot,\cdot)$ and $a_\nu(t;\cdot,\cdot)$ satisfy Gårding-type inequalities uniformly in t. Consequently, the initial-boundary value problems (4.13a),(4.13b) and (4.14a),(4.14b) have unique solutions $y \in W(0,T)$ and $y_\nu \in W_\nu(0,T), 1 \le \nu \le 2$, satisfying (4.16) (cf., e.g., Thm. 2.11 and section 2.6 in [1]). The assertions (4.15a),(4.15a) follow from standard regularity results for parabolic partial differential equations [16, 28, 38]. $\square$

Finally, the weak form of the optimal control problem can be stated as follows:

Find (y, u), where $y \in W(0,T), y|_{\Sigma_\nu} = y_\nu \in W_\nu(0,T), 1 \le \nu \le 2$, and $\mathbf{u} \in \mathbf{U}_{ad}$ such that

$$\inf_{y,u} J(y, \mathbf{u}) := \frac{1}{2} \int_0^T \int_\Omega |\nabla y - d|^2 dS dt, \tag{4.17a}$$

$$\text{subject to (4.13a),(4.13b) and (4.14a),(4.14b).} \tag{4.17b}$$

4.2 Existence and uniqueness of an optimal solution and first order necessary optimality conditions

In this section, we first prove the existence and uniqueness of an optimal solution of (4.17a),(4.17b) and then derive the first order necessary optimality conditions.

Theorem 4.3. *The optimal control problem* (4.17a),(4.17b) *admits a unique solution* $(y, \mathbf{u}) \in W(0,T) \times \mathbf{U}_{ad}$.

Proof. We prove the result with respect to the control-reduced formulation. To this end, we introduce $S : \mathbf{U}_{ad} \to W(0,T)$ and $S_\nu : \mathbf{U}_{ad} \to W_\nu(0,T), 1 \le \nu \le 2$, as the control-to-state maps which assign to an admissible control $\mathbf{u} \in \mathbf{U}_{ad}$ the unique solutions of the state equations (4.13a),(4.13b) and (4.14a),(4.14b). We further introduce the reduced objective functional

$$\hat{J}(\mathbf{u}) := \frac{1}{2} \int_0^T \int_\Omega |\nabla S(\mathbf{u}) - d|^2 dS dt. \tag{4.18}$$

The control-reduced formulation of (4.17a),(4.17b) reads:

$$\inf_{\mathbf{u} \in \mathbf{U}_{ad}} \hat{J}(\mathbf{u}), \tag{4.19a}$$

$$\text{such that } S(\mathbf{u}) \text{ and } S_\nu(\mathbf{u}), 1 \leq \nu \leq 2, \text{ satisfy (4.13a),(4.13b) and (4.14a),(4.14b).} \tag{4.19b}$$

We note that (4.19a),(4.19a) is equivalent to (4.17a),(4.17b). The existence and uniqueness of an optimal solution of (4.19a),(4.19a) follows by a standard minimizing sequence argument. □

For the derivation of the first order necessary optimality conditions we set

$$\mathbf{x} := (y, y_1, y_2, \mathbf{u}) \in X := W(0,T) \times W_1(0,T) \times W_2(0,T) \times \mathbb{R}^M,$$

and introduce Lagrange multipliers

$$\mathbf{z} := (p, p_1, p_2, q) \in Z := W(0,T) \times W_1(0,T) \times W_2(0,T) \times Q,$$

where

$$
\begin{aligned}
q &= ((q_{\Sigma_\nu})_{\nu=1}^4, (q_{K_{max}}^\nu)_{\nu=1}^2, (q_{K_{min}}^\nu)_{\nu=1}^2, q_{0,\Omega}, (q_{0,\Gamma_\nu})_{\nu=1}^2),\\
Q &:= \prod_{\nu=1}^4 L^2((0,T), H_\omega^{-1/2}(\Gamma_\nu)) \times \mathbb{R}^2 \times \mathbb{R}^2 \times L^2(\Omega) \times \prod_{\nu=1}^2 L^2(\Gamma_\nu).
\end{aligned}
$$

We define the Lagrangian $\mathcal{L} : X \times Z \to \mathbb{R}$ according to

$$
\begin{aligned}
\mathcal{L}(\mathbf{x},\mathbf{z})) := J(y,\mathbf{u}) &+ \int_0^T \Big(\langle \frac{\partial y}{\partial t}, p\rangle + a(t;y,p)\Big)\ dt + \sum_{\nu=1}^2 \int_0^T \Big(\langle \frac{\partial y_\nu}{\partial t}, p_\nu\rangle + a(t;y_\nu,p_\nu)\Big)\ dt \\
&+ \sum_{\nu=1}^4 \int_0^T \langle q_{\Sigma_\nu}, y_\nu - y|_{\Sigma_\nu}\rangle\ dt + \sum_{\nu=1}^2 \int_0^T \Big(q_{K_{max}}^\nu (u_M - y_\nu(K_{max})) - q_{K_{min}}^\nu y_\nu(K_{min}\Big) \\
&+ (y(0) - g(\mathbf{u}), q_{0,\Omega})_{L^2(\Omega)} + \sum_{\nu=1}^2 (y_\nu(0) - g(\mathbf{u}), q_{0,\Gamma_\nu})_{L^2(\Gamma_\nu)}.
\end{aligned}
$$

The first order necessary optimality conditions correspond to the conditions for a critical point of the Lagrangian:

$$
\frac{\partial \mathcal{L}}{\partial p}(\mathbf{x},\mathbf{z}) = 0, \quad \frac{\partial \mathcal{L}}{\partial p_\nu}(\mathbf{x},\mathbf{z}) = 0,\ 1 \le \nu \le 2, \tag{4.20a}
$$

$$
\frac{\partial \mathcal{L}}{\partial y}(\mathbf{x},\mathbf{z}) = 0, \quad \frac{\partial \mathcal{L}}{\partial y_\nu}(\mathbf{x},\mathbf{z}) = 0,\ 1 \le \nu \le 2, \quad \frac{\partial \mathcal{L}}{\partial q}(\mathbf{x},\mathbf{z}) = 0, \tag{4.20b}
$$

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{u}}(\mathbf{x},\mathbf{z}) \cdot (\mathbf{v} - \mathbf{u}) \ge 0,\ \mathbf{v} \in \mathbf{U}_{ad}. \tag{4.20c}
$$

Conditions (4.20a) clearly recover the state equations (4.11a)-(4.11c) and (4.11a)-(4.11c).

On the other hand, the conditions (4.20b) imply that

$$q_{\Sigma_\nu} = \gamma_{\Sigma_\nu}(n_{\Sigma_\nu} \cdot R_{\Sigma_\nu}(p_Q)) \ , \ 1 \le \nu \le 4, \tag{4.21a}$$

$$q_{K_{min},\nu} = R_{K_{min}}(p_{\Sigma_\nu}), \quad q_{K_{max},\nu} = R_{K_{max}}(p_{\Sigma_\nu}), \ 1 \le \nu \le 2, \tag{4.21b}$$

$$q_{0,\Omega} = \gamma_{0,\Omega}(p) \quad , \quad q_{0,\Sigma_\nu} = p_\nu(0) \ , \ 1 \le \nu \le 2, \tag{4.21c}$$

and

$$y|_{\Sigma_\nu} = y_\nu, \ 1 \le \nu \le 4, \tag{4.22a}$$

$$y_\nu(K_{min}) = 0 \quad , \quad y_\nu(K_{max}) = u_M, \ 1 \le \nu \le 2, \tag{4.22b}$$

$$y(\cdot, 0) = g(\mathbf{u}), \ y_\nu(\cdot, 0) = g(\mathbf{u})|_{\Sigma_\nu} \ , \ 1 \le \nu \le 2. \tag{4.22c}$$

Moreover, it follows that $p_\nu, 1 \le \nu \le 2$, is the weak solution of

$$-\frac{\partial p_\nu}{\partial t} - A_\nu^*(t) p_\nu = \gamma_{\Sigma_\nu}(n_{\Sigma_\nu} \cdot R_{\Sigma_\nu}(p)) \quad \text{in } \Gamma_\nu, \tag{4.23a}$$

$$R_{K_{min}}(p_\nu) = R_{K_{max}}(p_\nu) = 0, \tag{4.23b}$$

$$p_\nu(\cdot, T) = 0 \quad \text{in } \Gamma_\nu. \tag{4.23c}$$

where $A_\nu^*(t)$ is the adjoint of $A_\nu(t)$ and $R_{\Sigma_\nu}(p)), 1 \le \nu \le 4$, as well as $R_K(p_\nu), K \in \{K_{min}, K_{max}\}$ are given by

$$R_{\Sigma_\nu}(p) = (R^{(1)}_{\Sigma_\nu}(p), R^{(2)}_{\Sigma_4}(p))^T, \quad R^{(k)}_{\Sigma_\nu}(p) := S_k\Big(\frac{1}{2}\sum_{\ell=1}^{2} \xi_{k\ell} S_\ell \frac{\partial p}{\partial S_\ell} - rp\Big), \ 1 \le k \le 2,$$

$$R_K(p_\nu) := \frac{1}{2}\sigma_\nu^2 S_K^2 \frac{\partial p_\nu}{\partial S_\nu}(K) - rS_K p_\nu(K), \ 1 \le \nu \le 2, \ K \in \{K_{min}, K_{max}\}.$$

Since $(n_{\Sigma_\nu} \cdot R_{\Sigma_\nu}(p))|_{\Sigma_\nu} = 0, \ 1 \le \nu \le 2$, it follows that $p_\nu = 0, 1 \le \nu \le 2$. Moreover, we deduce that p is the weak solution of

$$-\frac{\partial p}{\partial t} - A^*(t)p = \; -\nabla\cdot(\nabla y - d) \quad \text{in } Q, \tag{4.24a}$$

$$p = 0 \quad \text{on } \Sigma, \tag{4.24b}$$

$$p(\cdot,T) = 0 \quad \text{in } \Omega, \tag{4.24c}$$

where $A^*(t)$ is the second order elliptic differential operator adjoint to $A(t)$. Finally, observing (4.8) and $y_4 = u_M$ as well as the regularity results of Theorem 4.2, the condition (4.22c) implies the variational inequality

$$\Big(\int\limits_0^T \Big(\gamma_{\Sigma_4}(n_{\Sigma_4}\cdot R_{\Sigma_4}(p))\Big)\, dt\; \mathbf{e}_M - g^*_{\mathbf{u}}(\mathbf{u})p(0)\Big)\cdot(\mathbf{v}-\mathbf{u}) \geq 0,\;\; \mathbf{v}\in\mathbf{U}_{ad}, \tag{4.25}$$

where $\mathbf{e}_M \in \mathbb{R}^M$ is the M-th unit vector and $g^*_{\mathbf{u}}(\mathbf{u}) \in \mathcal{L}(L^2(\Omega),\mathbb{R}^M)$ is the adjoint of the Fréchet derivative of g at $\mathbf{u}\in\mathbf{U}_{ad}$.

Summarizing the previous findings, we have the following result:

Theorem 4.4. *Assume that $(y,u) \in W(0,T)\times\mathbf{U}_{ad}$ is the optimal solution of* (4.17a),(4.17b). *Then, there exists*

$$p \in W_0(0,T) := H^1((0,T),H^{-1}_\omega(\Omega)) \cap L^2((0,T),H^1_{\omega,0}(\Omega)), \tag{4.26}$$

such that p is the weak solution of the adjoint problem (4.24a)-(4.24c) *and the variational inequality* (4.25) *holds true.*

4.3 Discretization of the Optimal Control Problem

4.3.1 Semi-Discretization in Space

More details can be found in [5, 7]. We discretize the parabolic problems (4.13a),(4.13b) and (4.14a),(4.14b) in space by conforming P1 finite elements. To this end, we consider a family of shape-regular simplicial triangulations $\mathcal{T}_h(\Omega)$ of Ω that are assumed to align with $\Gamma_j, 1 \leq j \leq 4$, in the sense that these triangulations also generate triangulations $\mathcal{T}_h(\Gamma_j)$ of $\Gamma_j, 1 \leq j \leq 4$. Using standard notation from the finite element analysis, we refer to $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D)$, $D \subseteq \overline{\Omega}$, as the sets of vertices and edges in $D \subseteq \overline{\Omega}$. We denote by h_T and $|T|$ the diameter and area of an element $T \in \mathcal{T}_h^{(m)}(\Omega)$. For $D \subset \overline{\Omega}$, we refer to $P_k(D), k \in \mathbb{N}_0$, as the linear spaces of polynomials of degree $\leq k$ on D.

We define V_h as the finite element space of continuous P1 finite elements associated with the triangulation $\mathcal{T}_h(\Omega)$, i.e., $V_h := \{v_h \in C(\bar{\Omega}) \mid v_h|_T \in P_1(T), T \in \mathcal{T}_h(\Omega)\}$, and we refer to $V_{h,0} := V_h \cap C_0(\bar{\Omega})$ as the associated finite element space of functions vanishing on the boundary Γ. Likewise, we define $V_{h,\nu}, 1 \leq \nu \leq 2$, as the finite element spaces of continuous P1 finite elements associated with the triangulations $\mathcal{T}_h(\Gamma_\nu)$ attaining the values 0 at $S_\nu = K_{min}$ and u_M at $S_\nu = K_{max}$, i.e., $V_{h,\nu} := \{v_h \in C(\bar{\Gamma}_\nu) \mid v_h|_T \in P_1(T), T \in \mathcal{T}_h(\Gamma_\nu),\ v_h(K_{min}) = 0,\ v_h(K_{max}) = u_M\}$, and we define $V_{h,\nu,0}$ in the same way, but replacing u_M with 0.

The semi-discrete approximation of (4.14a),(4.14b) requires the computation of $y_{h,\nu} \in C^1([0,T], V_{h,\nu}), 1 \leq \nu \leq 2$, such that

$$(\frac{dy_{h,\nu}}{dt}, v_h)_{L^2(\Gamma_\nu)} + a(t; y_{h,\nu}, v_h) = 0\ ,\ v_h \in V_{h,\nu,0}, \tag{4.27a}$$

$$(y_{h,\nu}(\cdot,0), v_h)_{L^2(\Gamma_\nu)} = (g(\mathbf{u}), v_h)_{L^2(\Gamma_\nu)}\ ,\ v_h \in V_{h,\nu}. \tag{4.27b}$$

The semi-discrete approximation of (4.13a),(4.13b) requires the computation of $y_h \in C^1([0,T],V_h)$ with $y_h(\cdot,t)|_{\Gamma_\nu} = y_{h,\nu}(\cdot,t), 1 \le \nu \le 2$, and $y_h(\cdot,t)|_{\Gamma_3} = 0, y_h(\cdot,t)|_{\Gamma_4} = u_M$, such that

$$(\frac{dy_h}{dt}, v_h)_{L^2(\Omega)} + a(t; y_h, v_h) = 0 \ , \ v_h \in V_{h,0}, \tag{4.28a}$$

$$(y_h(\cdot,0), v_h)_{L^2(\Omega)} = (g(\mathbf{u}), v_h)_{L^2(\Omega)} \ , \ v_h \in V_h. \tag{4.28b}$$

The semi-discrete optimal control problems reads: Find $(y_h, \mathbf{u})$ such that

$$\inf_{y_h,\mathbf{u}} J_h(y_h,\mathbf{u}) := \frac{1}{2} \int_0^T \sum_{K \in \mathcal{T}_h(\Omega)} \|\nabla(y_h(\cdot,t) - d\|^2_{L^2(K)} \ dt, \tag{4.29a}$$

$$\text{subject to } (4.27a), (4.27b), (4.28a), (4.28b) \text{ and } (4.9). \tag{4.29b}$$

4.3.2 Algebraic formulation of the semi-discretized problem

We derive the algebraic formulation of (4.29a),(4.29b) in terms of associated mass and stiffness matrices $\mathbf{M} \in \mathbb{R}^{N\times N}, \mathbf{A}(t) \in \mathbb{R}^{N\times N}$, input matrices $\mathbf{B}(t), \mathbf{G} \in \mathbb{R}^{N\times M}$, and observation matrices $\mathbf{C}, \mathbf{D}^{(k)} \in \mathbb{R}^{N_Q\times N_Q}, 1 \le k \le 2$. We note that $N := N_Q + N_{\Gamma_1} + N_{\Gamma_2}$, where N_Q, N_{Γ_ν} stand for the number of nodal points in $\mathcal{N}_h(\Omega)$ and $\mathcal{N}_h(\Gamma_\nu), 1 \le \nu \le 2$, respectively. In the sequel, we refer to $\psi^i_\Omega, 1 \le i \le N_Q$, and $\psi^i_{\Gamma_\nu}, 1 \le i \le N_{\Gamma_\nu}$, as the nodal basis functions associated with the nodal points in $\mathcal{N}_h(\Gamma_1), \mathcal{N}_h(\Gamma_2)$, and $\mathcal{N}_h(\Gamma_4)$.

Mass matrix: The mass matrix $\mathbf{M} \in \mathbb{R}^{N\times N}$ is a block-structured matrix of the form

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_\Omega & \mathbf{M}_{\Omega\Gamma_1} & \mathbf{M}_{\Omega\Gamma_2} \\ 0 & \mathbf{M}_{\Gamma_1} & 0 \\ 0 & 0 & \mathbf{M}_{\Gamma_2} \end{pmatrix},$$

Here, $\mathbf{M}_\Omega \in \mathbb{R}^{N_\Omega\times N_\Omega}, \mathbf{M}_{\Omega\Gamma_\nu} \in \mathbb{R}^{N_\Omega\times N_{\Gamma_\nu}}$ and $\mathbf{M}_{\Gamma_\nu\Gamma_\nu}, 1 \le \nu \le 2$, are the submatrices

$$\begin{aligned}
(\mathbf{M}_\Omega)_{ij} &:= (\psi_\Omega^j, \psi_\Omega^i)_{L^2(\Omega)} \ , \ 1 \le i,j \le N_\Omega, \\
(\mathbf{M}_{\Omega\Gamma_\nu})_{ij} &:= (\psi_\Omega^j, \psi_\Omega^i)_{L^2(\Omega)} \ , \ 1 \le i \le N_\Omega, \ 1 \le j \le N_{\Gamma_\nu} \ , \ 1 \le \nu \le 2, \\
(\mathbf{M}_{\Gamma_\nu})_{ij} &:= (\psi_{\Gamma_\nu}^j, \psi_{\Gamma_\nu}^i)_{L^2(\Gamma_\nu)} \ , \ 1 \le i,j \le N_{\Gamma_\nu} \ , \ 1 \le \nu \le 2,
\end{aligned}$$

Stiffness matrix: The stiffness matrix is a block-structured matrix of the form

$$\mathbf{A}(t) = \begin{pmatrix} \mathbf{A}_\Omega(t) & \mathbf{A}_{\Omega\Gamma_1}(t) & \mathbf{A}_{\Omega\Gamma_2}(t) \\ 0 & \mathbf{A}_{\Gamma_1}(t) & 0 \\ 0 & 0 & \mathbf{A}_{\Gamma_2}(t) \end{pmatrix}, \ t \in (0,T].$$

Here, the submatrices $\mathbf{A}_\Omega(t) \in \mathbb{R}^{N_\Omega\times N_\Omega}, \mathbf{A}_{\Omega\Gamma_\nu}(t) \in \mathbb{R}^{N_\Omega\times N_{\Gamma_\nu}}, \mathbf{A}_{\Gamma_\nu}(t) \in \mathbb{R}^{N_{\Gamma_\nu}\times N_{\Gamma_\nu}}$ are given by

$$\begin{aligned}
(\mathbf{A}_\Omega(t))_{ij} &:= a(t; \psi_\Omega^j, \psi_\Omega^i) \ , \ 1 \le i,j \le N_\Omega, \\
(\mathbf{A}_{\Omega\Gamma_\nu}(t))_{ij} &:= a(t; \psi_{\Gamma_\nu}^j, \psi_\Omega^i) \ , \ 1 \le i \le N_\Omega \ , \ 1 \le j \le N_{\Gamma_\nu} \ , \ 1 \le \nu \le 2, \\
(\mathbf{A}_{\Gamma_\nu}(t))_{ij} &:= a(t; \psi_{\Gamma_\nu}^j, \psi_{\Gamma_\nu}^i) \ , \ 1 \le i,j \le N_{\Gamma_\nu} \ , \ 1 \le \nu \le 2.
\end{aligned}$$

Input matrices: The input matrix $\mathbf{B}(t) \in \mathbb{R}^{N\times M}$ describes the input from the controls on the boundaries. It is of the form

$$\mathbf{B}(t) = (0\ \mathbf{B}_M(t))\ ,\ 0 \in \mathbb{R}^{N\times(M-1)},\quad \mathbf{B}_M(t) = (\mathbf{B}_{M,\Omega}(t), \mathbf{B}_{M,\Gamma_1}(t), \mathbf{B}_{M,\Gamma_2}(t))^T\ ,\ t \in (0,T],$$

The submatrices $\mathbf{B}_{M,\Omega}(t) \in \mathbb{R}^{N_Q\times 1}$ and $\mathbf{B}_{M,\Gamma_\nu}(t) \in \mathbb{R}^{N_{\Gamma_\nu}\times 1}, 1 \le \nu \le 2$, are given by

$$\begin{aligned}
(\mathbf{B}_{M,\Omega}(t))_i &:= -\sum_{j=1}^{N_{\Gamma_4}} a(t; \psi^j_{\Gamma_4}, \psi^i_\Omega)\ ,\ 1 \le i \le N_\Omega,\\
(\mathbf{B}_{M,\Gamma_\nu}(t))_i &:= -a(t; \psi_{\Gamma_4}^{N^{(\nu)}_{\Gamma_4}}, \psi^i_{\Gamma_\nu})\ ,\ 1 \le i \le N_{\Gamma_\nu}\ ,\ 1 \le \nu \le 2,
\end{aligned}$$

where $N^{(\nu)}_{\Gamma_4} := (2-\nu) + (\nu - 1)N_{\Gamma_4}, 1 \le \nu \le 2$.

The input matrix $\mathbf{G} \in \mathbb{R}^{N\times M}$ describes the input from the initial control. It has the form

$$\mathbf{G} = (\mathbf{G}_\Omega, \mathbf{G}_{\Gamma_1}, \mathbf{G}_{\Gamma_2})^T,$$

where the submatrices $\mathbf{G}_\Omega \in \mathbb{R}^{N_\Omega\times M}$ and $\mathbf{G}_{\Gamma_\nu} \in \mathbb{R}^{N_{\Gamma_\nu}\times M}, 1 \le \nu \le 2$, are given by

$$\begin{aligned}
(\mathbf{G}_\Omega)_{ij} &:= \int_{\Omega_j} g_2^{(j)}(S)\psi^i_\Omega(S)dS + \int_{\Omega_{j+1}} g_1^{(j+1)}(S)\psi^i_\Omega(S)dS,\\
(\mathbf{G}_{\Gamma_\nu})_{ij} &:= \int_{K_{j-1}}^{K_j} g_2^{(j)}(S_\nu)\psi^i_{\Gamma_\nu}(S_\nu)dS_\nu + \int_{K_j}^{K_{j+1}} g_1^{(j+1)}(S_\nu)\psi^i_{\Gamma_\nu}(S_\nu)dS_\nu\ ,\ 1 \le \nu \le 2.
\end{aligned}$$

Observation matrices: The observation matrices $\mathbf{C}$ and $\mathbf{D}^{(k)}, 1 \le k \le 2$, stem from the semi-discretization in space of the objective functional. In particular, $\mathbf{C} \in \mathbb{R}^{N_\Omega\times N_\Omega}$ has the entries

$$(\mathbf{C})_{ij} := \sum_{T \in \mathcal{T}_h(\Omega)} \int_T \nabla \psi_\Omega^j \cdot \nabla \psi_\Omega^i dS \ , \ 1 \le i,j \le N_\Omega,$$

whereas the entries of $\mathbf{D}^{(k)} \in \mathbb{R}^{N_\Omega \times N_\Omega}$ are given by

$$(\mathbf{D}^{(k)})_{ij} := \sum_{T \in \mathcal{T}_h(\Omega)} \int_T \frac{\partial \psi_\Omega^j}{\partial S_k} \psi_\Omega^i dS \ , \ 1 \le i,j \le N_\Omega \ , \ 1 \le k \le 2.$$

Algebraic formulation of the semi-discrete optimal control problem:

Find $y \in C^1([0,T],\mathbb{R}^N), y = (y_Q, y_1, y_2)^T$, and $\mathbf{u} \in \mathbf{U}_{ad}$, such that

$$\inf_{y,\mathbf{u}} J(y,\mathbf{u}) := \frac{1}{2} \int_0^T \Big(y_Q^T \mathbf{C} y_Q - 2 \sum_{k=1}^2 d_k^T \mathbf{D}^{(k)} y_Q + \sum_{k=1}^2 d_k^T \mathbf{M}_\Omega d_k \Big) \ dt, \tag{4.30a}$$

subject to

$$\mathbf{M} \ \frac{dy}{dt} + \mathbf{A}(t) y = \mathbf{B}\mathbf{u} \ , \ t \in [0,T], \tag{4.30b}$$

$$\mathbf{M} y(0) = \mathbf{G}\mathbf{u}. \tag{4.30c}$$

Existence and uniqueness of a solution:

The existence and uniqueness of an optimal solution can be shown along the same lines of proof as in the continuous regime.

First order necessary optimality conditions:

For the derivation of the first order necessary optimality conditions we set

$$\mathbf{x} := (y,\mathbf{u}) \in \mathbf{X} := C^1([0,T],\mathbb{R}^{N_Q}) \times \mathbb{R}^M$$

and introduce Lagrange multipliers

$$
\begin{aligned}
&\mathbf{z} := (p, p_1, p_2, q_0) \in \mathbf{Z}, \\
&\mathbf{Z} := C^1([0,T], \mathbb{R}^N_Q) \times \prod_{\nu=1}^{2} C^1([0,T], \mathbb{R}^N_{\Gamma_\nu}) \times \mathbb{R}^{N_Q}.
\end{aligned}
$$

The Lagrangian $\mathcal{L} : \mathbf{X} \times \mathbf{Z} \to \mathbb{R}$ is given by

$$
\mathcal{L}(\mathbf{x}, \mathbf{z}) := J(y, \mathbf{u}) + \int_0^T p \cdot (\mathbf{M} \, \frac{dy}{dt} + \mathbf{A}(t)y - \mathbf{B}\mathbf{u}) \; dt + q_0 \cdot (\mathbf{M}y(0) - \mathbf{G}\mathbf{u}),
$$

and the optimality conditions read

$$
\frac{\partial \mathcal{L}}{\partial y}(\mathbf{x}, \mathbf{z}) = 0, \tag{4.31a}
$$

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{u}}(\mathbf{x}, \mathbf{z}) \cdot (\mathbf{v} - \mathbf{u}) \geq 0, \quad \mathbf{v} \in \mathbf{U}_{ad}, \tag{4.31b}
$$

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{z}}(\mathbf{x}, \mathbf{z}) = 0. \tag{4.31c}
$$

In particular, the optimality condition (4.31a) reveals that p solves the adjoint system

$$
\mathbf{M}_\Omega \, \frac{dp}{dt} - \mathbf{A}_\Omega(t)^T p = \; - \mathbf{C}_\Omega y_Q + \sum_{k=1}^{2} (\mathbf{D}_\Omega^{(k)})^T d_k, \; t \in [0,T], \tag{4.32a}
$$

$$
\mathbf{M}p(T) = 0, \tag{4.32b}
$$

and that $p_\nu = 0, 1 \leq \nu \leq 2$ as well as $q_0 = p(0)$.

On the other hand, the optimality condition (4.31b) gives rise to

$$
\Big(- \mathbf{G}_\Omega^T p(0) - \int_0^T \mathbf{B}_\Omega(t)^T p \; dt \Big) \cdot (\mathbf{v} - \mathbf{u}) \geq 0 \quad , \quad \mathbf{v} \in \mathbf{U}_{ad}. \tag{4.33}
$$

We have thus shown the following result:

Theorem 4.5. *The semi-discrete optimization problem* (4.30a)-(4.30c) *admits a unique solution. If* $y \in C^1([0,T], \mathbb{R}^N), \mathbf{u} \in \mathbf{U}_{ad}$ *is the optimal solution, there exists* $p \in C^1([0,T], \mathbb{R}^N_\Omega)$ *such that* p *satisfies the adjoint system* (4.32a),(4.32b) *and* $p, \mathbf{u}$ *are related by the variational inequality* (4.33).

4.3.3 Implicit time stepping

The discretization in time of the semi-discrete optimal control problem (4.30a)-(4.30c) is done by the implicit time stepping with respect to a partition $0 =: t_0 < t_1 < \cdots < t_R := T/R, R \in \mathbb{N}$, of the time interval $[0,T]$ with step lengths $\Delta t_r := t_r - t_{r-1}, 1 \leq r \leq R$. In particular, the objective functional (4.30a) is split into the sum over the subintervals (t_{r-1}, t_r) and the corresponding integrals are approximated by the quadrature formula $\int_{t_{r-1}}^{t_r} v dt \approx \Delta t_r v(t_r)$, whereas the ordinary differential equation (4.30b) is approximated by the implicit Euler scheme. We denote by

$$y^r = (y^r_Q, y^r_{\Sigma_1}, y^r_{\Sigma_1})^T$$

approximations of $y = (y_Q, y_{\Sigma_1}, y_{\Sigma_1})^T$ at $t_r, 0 \leq r \leq R$, and we set

$$\mathbf{y} := (y^0, \cdots, y^R)^T, \mathbf{y}_Q := (y^0_Q, \cdots, y^R_Q)^T, \mathbf{y}_{\Sigma_\nu} := (y^0_{\Sigma_\nu}, \cdots, y^R_{\Sigma_\nu})^T, 1 \leq \nu \leq 2.$$

The fully discrete optimal control problem can be stated as follows:

Find $(\mathbf{y}, u) \in \mathbb{R}^{(R+1)N} \times \mathbf{U}_{ad}$ such that

$$\inf_{\mathbf{y},\mathbf{u}} J(\mathbf{y},\mathbf{u}) := \frac{1}{2}\sum_{r=1}^{R} \Delta t_r \Big((y_Q^r)^T \mathbf{C}_\Omega y_Q^r - 2\sum_{k=1}^{2} d_k^T \mathbf{D}_\Omega^{(k)} y_Q^r + \sum_{k=1}^{2} d_k^T \mathbf{M}_\Omega d_k \Big), \tag{4.34a}$$

subject to

$$\mathbf{M}y^r + \Delta t_r \mathbf{A}(t_r) y^r = \Delta t_r \mathbf{B}\mathbf{u} + \mathbf{M}y^{r-1}, \; 1 \le r \le R, \tag{4.34b}$$

$$\mathbf{M}y^0 = \mathbf{G}\mathbf{u}. \tag{4.34c}$$

The existence and uniqueness of an optimal solution follows as in the previous subsection 4.3.2, and the optimality conditions can be derived as well in much the same manner. In particular, there exists an adjoint state $\mathbf{p} = (p^0, \cdots, p^R)^T \in \mathbb{R}^{(R+1)N_Q}$ such that

$$\mathbf{M}_\Omega p^{r-1} + \Delta t_r \mathbf{A}_\Omega(t_{r-1})^T p^{r-1} = \mathbf{M}_\Omega p^r + \Delta t_r (\mathbf{C}_\Omega y_Q^r + \sum_{k=1}^{2} (\mathbf{D}_\Omega^{(k)})^T d_k), \tag{4.35a}$$

$$\mathbf{M}_\Omega p^R = 0. \tag{4.35b}$$

Moreover, the following variational inequality holds true

$$\Big(-\mathbf{G}_\Omega^T p^0 - \sum_{r=0}^{R-1} \Delta t_{r+1} \mathbf{B}(t_r)^T p^r \Big) \cdot (\mathbf{v} - \mathbf{u}) \ge 0 \quad , \quad \mathbf{v} \in \mathbf{U}_{ad}. \tag{4.36}$$

Summarizing, we have the following result:

Theorem 4.6. *The fully discrete optimization problem* (4.34a)-(4.34c) *admits a unique solution. If* $\mathbf{y} \in \mathbb{R}^{(R+1)N}, \mathbf{u} \in \mathbf{U}_{ad}$ *is the optimal solution, there exists* $\mathbf{p} = (p^0, \cdots, p^R)^T \in \mathbb{R}^{(R+1)N_Q}$ *such that the adjoint system* (4.35a),(4.35b) *holds true and the variational inequality* (4.36) *is satisfied.*

Chapter 5

Numerical Results

In the first section 5.1 of this chapter, we apply the projected gradient method with line search(cf., e.g., [3]) as a solver for the fully discretized optimal control problem, whereas the subsequent section 5.2 is devoted to a documentation of computational results.

5.1 Projected gradient method with line search

The control-reduced form of the fully discrete optimal control problem (4.34a)-(4.34c) is given by

$$\inf_{\mathbf{u}\in\mathbf{U}_{ad}} \hat{J}(\mathbf{u}), \quad \hat{J}(\mathbf{u}) := J(S(\mathbf{u}),\mathbf{u}), \tag{5.1}$$

where $S : \mathbf{U}_{ad} \to \mathbb{R}^{(R+1)N}$ stands for the control-to-state map which assigns to an admissible control $\mathbf{u} \in \mathbf{U}_{ad}$ the solution $\mathbf{y} \in \mathbb{R}^{(R+1)N}$ of the discrete state equation (4.34a),(4.34b).

According to Theorem 4.6, the gradient of the control-reduced objective functional reads

$$\nabla \hat{J}(\mathbf{u}) = -\mathbf{G}_{\Omega}^{T} p_{Q}^{0} - \sum_{r=0}^{R-1} \Delta t_{r+1} \mathbf{B}_{\Omega}(t_r)^T p^r. \tag{5.2}$$

Given an initial control $\mathbf{u}^{(0)} \in \mathbf{U}_{ad}$, we solve (4.34a)-(4.34c) by the projected gradient method with Armijo line search (cf., e.g., [25, 35])

$$\mathbf{u}^{(\ell+1)} = \mathbf{u}^{(\ell)} - \alpha_\ell \nabla \hat{J}(\mathbf{u}^{(\ell)})), \ \ell \geq 0. \tag{5.3}$$

Here, α_ℓ is the step length chosen such that $\mathbf{u}^{(\ell+1)}$ is feasible, i.e., $\mathbf{u}^{(\ell+1)} \in \mathbf{U}_{ad}$, and that the Wolfe conditions

$$\hat{J}(\mathbf{u}^{(\ell)} - \alpha_\ell \nabla \hat{J}(\mathbf{u}^{(\ell)})) \leq \hat{J}(\mathbf{u}^{(\ell)}) - c_1 \alpha_\ell \|\nabla \hat{J}(\mathbf{u}^{(\ell)})\|^2, \tag{5.4a}$$

$$\nabla \hat{J}(\mathbf{u}^{(\ell)})^T \nabla \hat{J}(\mathbf{u}^{(\ell)} - \alpha_\ell \nabla \hat{J}(\mathbf{u}^{(\ell)})) \leq c_2 \|\nabla \hat{J}(\mathbf{u}^{(\ell)})\|^2, \tag{5.4b}$$

are satisfied, where $0 < c_1 \ll c_2 < 1$. We note that (5.4a) is called the Armijo rule [3], whereas (5.4b) is referred to as the curvature condition.

Comparably, we solve (4.34a)-(4.34c) by the projected gradient method with Backtracking line search with algorithm as follows:

given a descent direction Δx for f at $x \in dom f$, $\alpha \in (0, 0.5)$, $\beta \in (0, 1)$.

$t := 1$.

while $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^T \Delta x$, $t := \beta t$

5.2 Numerical results

We study the performance of the projected gradient method in case of a fixed maturity time and strike, fixed lower and upper barriers, and a fixed number of cash settlements (controls), but various values of the desired Delta, the interest rate, and the volatilities of the underlying assets. Table 5.1 contains the data that remain fixed for all numerical experiments.

For discretization in space, we have chosen a simplicial triangulation $\mathcal{T}_h(\Omega)$ of the trapezoidal domain Ω with $h := \max\{\mathrm{diam}(T) \mid T \in \mathcal{T}_h(\Omega)\} = 5.0$ for both the state and the adjoint state. On the other hand, for discretization in time we have used a uniform time step of $\Delta t = 0.01$. The projected gradient method with line search has been initialized with an initial control $u_0 = (0, 50, 0, 50, 0)^T$ and has been stopped when the projected gradient became smaller than $TOL := 1.0E-06$.

Parameter	**Notation**	**Value**
M	Number of controls	5
K_{min}	Lower Barrier	50
K_{max}	Upper Barrier	150
K	Strike	100
T	Maturity	1
ρ	Correlation between assets	-0.5
$u_{i,min}$	Lower bound on the controls	0.0
$u_{i,max}$	Upper bound on the controls	50.0

Table 5.1: Data of the optimal control problem that remain fixed for all experiments

Experiments 1-6. In the first example, we study the impact of different desired Deltas, different interest rates, and different volatilities, whereas the respective other data are kept fixed. Table 5.2 contains the desired delta, interest rate and volatilities for the two underlying assets used in the experiments 1-6.

Parameter	Exp. 1	Exp. 2	Exp. 3	Exp. 4	Exp. 5	Exp. 6
d	(0.1,0.4)	(0.4,0.1)	(0.3,0.3)	(0.3,0.3)	(0.3,0.3)	(0.3,0.3)
r	0.04	0.04	0.02	0.10	0.04	0.04
σ_1	0.25	0.25	0.25	0.25	0.10	0.40
σ_2	0.25	0.25	0.25	0.25	0.40	0.10

Table 5.2: Values of the desired Delta $d = (d_1, d_2)$, the interest rate r, and the volatilities σ_1, σ_2 of the underling assets in Experiments 1-6.

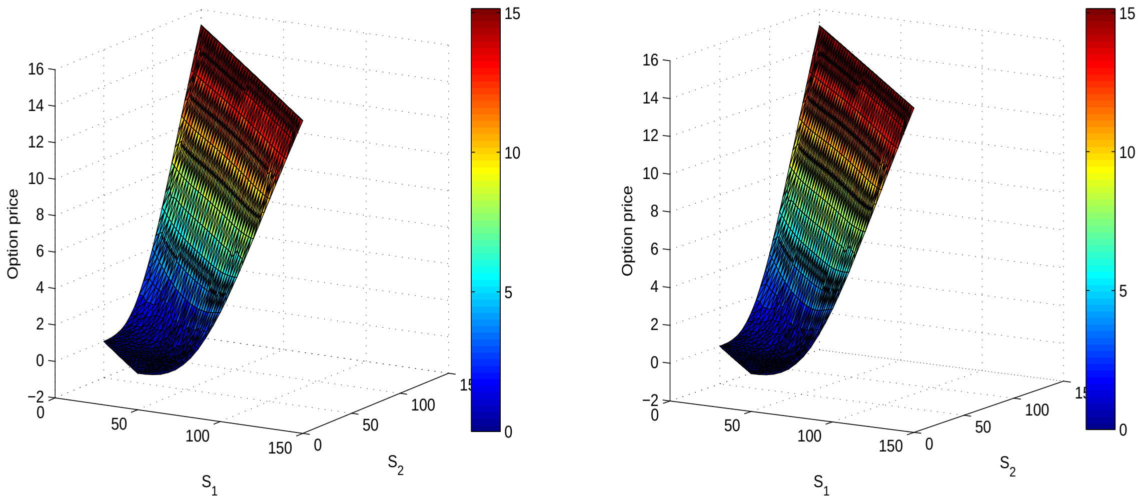

Figure 5.1: Option price at maturity for Exp. 1 (left) and Exp. 2 (right).

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	2.105e+03	5.237e+01	10	2.242e+02	2.161e+00
20	1.080e+02	4.272e-03	30	1.080e+02	9.623e-06
34	1.080e+02	3.238e-07			

Table 5.3: Experiment 1 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	1.226e+03	3.566e+01	10	1.130e+02	1.015e-01
19	1.080e+02	6.811e-05			

Table 5.4: Experiment 1 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	2.227e+03	5.603e+01	10	9.218e+01	4.997e+00
20	1.389e+02	2.446e-02	30	1.388e+02	4.053e-06
33	1.388e+02	1.886e-07			

Table 5.5: Experiment 2 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	1.799e+02	9.338e+00	10	1.389e+02	2.174e-02
13	1.388e+02	3.209e-04			

Table 5.6: Experiment 2 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

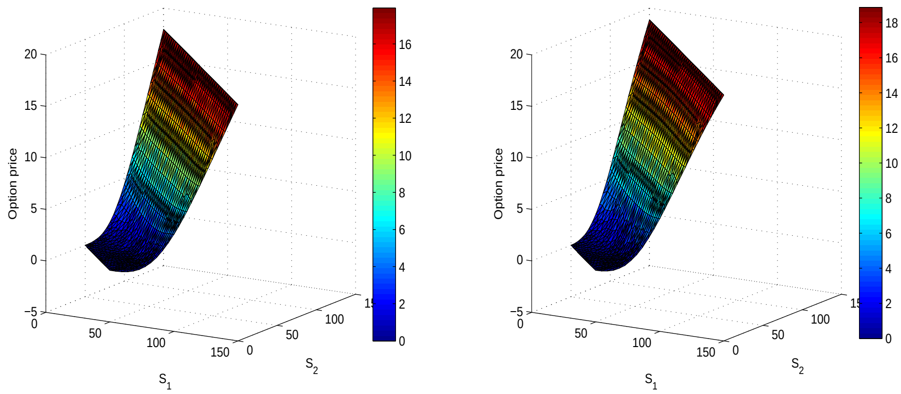

Figure 5.2: Option price at maturity for Exp. 3 (left) and Exp. 4 (right).

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	2.931e+03	5.673e+01	10	6.741e+01	3.083e+00
20	1.594e+01	3.766e-03	30	1.595e+01	6.460e-06
34	1.595e+01	6.990e-07			

Table 5.7: Experiment 3 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	6.828e+02	1.081e+01	10	1.596e+01	1.013e-02
20	1.595e+01	6.836e-04			

Table 5.8: Experiment 3 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	2.353e+03	4.638e+01	10	2.870e+01	1.404e+00
20	1.708e+01	2.405e-03	30	1.708e+01	7.917e-06
33	1.708e+01	5.799e-07			

Table 5.9: Experiment 4 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	1.131e+02	6.232e+00	10	1.709e+01	1.225e-02
14	1.708e+01	9.361e-05			

Table 5.10: Experiment 4 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

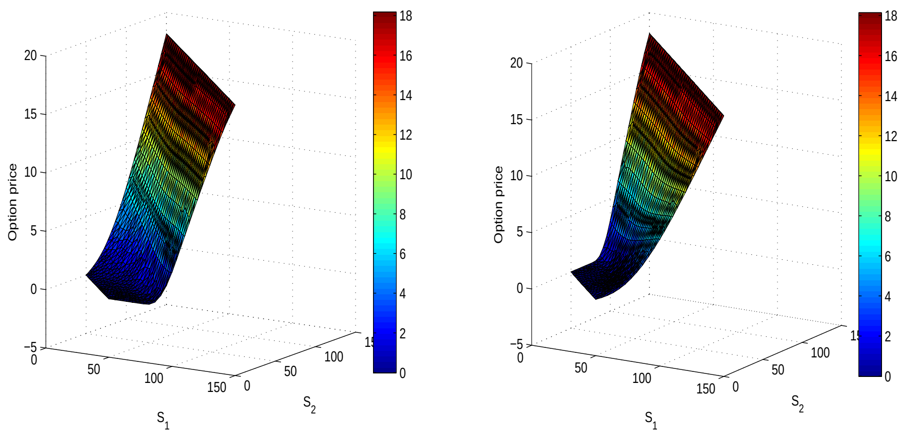

Figure 5.3: Option price at maturity for Exp. 5 (left) and Exp. 6 (right).

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	2.971e+03	5.971e+01	10	5.723e+01	2.085e+00
20	1.970e+01	6.970e-04	30	1.970e+01	8.502e-07

Table 5.11: Experiment 5 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	1.481e+003	4.760e+01	10	4.101e+01	1.806e+00
20	1.970e+001	1.058e-03	29	1.970e+01	1.870e-05

Table 5.12: Experiment 5 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	3.366e+03	5.932e+01	10	5.380e+01	2.115e+00
20	1.692e+01	9.243e-04	30	1.692e+01	4.625e-07

Table 5.13: Experiment 6 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	9.510e+02	3.468e+01	10	1.751e+01	2.924e-01
17	1.692e+01	7.524e-06			

Table 5.14: Experiment 6 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

Tables 5.3-5.14 contain a documentation of the convergence history of the projected gradient algorithm with Armijo line search and Back-tracking line search. Here, ℓ stands for the iteration number, $J_{red}(u^{(\ell)})$ is the corresponding value of the objective functional, and $\|\nabla J_{red}(u^{(\ell)})\|$ refers to the norm of the gradient. As a termination criterion for the iteration, we have used $\|\nabla J_{red}(u^{(\ell)})\| < TOL := 1.0E - 06$. When we use projected gradient method with Back-tracking line search, we need to find appropriate initials to get the good convergence rate, however, we can pick the initials randomly for the Armijo line search to achieve better results.

As far as the impact of different desired Deltas is concerned, in Figure 5.1 we observe that the option price with respect to S_2 is a bit higher in Exp. 1 than in Exp. 2 contrary to the price with respect to S_1 which is lower in Exp. 1 than in Exp. 2.
With regard to the influence of different interest rates, Figure 5.2 reveals that the option price is higher with respect to both S_1 and S_2 for higher interest rates.
Finally, the impact of different volatilities is displayed in Figure 5.3. In Exp. 5 (σ_1 is lower than σ_2), the option price increases more rapidly in S_1 than in S_2, whereas in Exp. 6 (values of σ_1 and σ_2 exchanged) we observe the opposite behavior.

Experiments 7-12. The second set of experiments deals with the case of time-varying interest rate r and space-varying volatilities σ_1, σ_2:

$$r(t) = r_1 \cdot t + r_2 \cdot (1 - t),$$

$$\sigma_1(S,t) = \widehat{\sigma_1} \cdot ((S_1 + S_2 - 100)/50)^2,$$

$$\sigma_2(S,t) = \widehat{\sigma_2} \cdot ((S_1 + S_2 - 100)/50)^2.$$

The values of the desired Deltas $d = (d_1, d_2)$ and of the coefficients r_1, r_2 in $r(t)$ as well as the coefficients $\widehat{\sigma_1}$, $\widehat{\sigma_2}$ in $\sigma_1(S,t)$, $\sigma_2(S,t)$ are given in the following table.

Parameter	Exp. 7	Exp. 8	Exp. 9	Exp. 10	Exp. 11	Exp. 12
d	(0.1,0.4)	(0.4,0.1)	(0.3,0.3)	(0.3,0.3)	(0.3,0.3)	(0.3,0.3)
r_1	0.03	0.03	0.02	0.08	0.03	0.03
r_2	0.07	0.07	0.08	0.02	0.07	0.07
$\widehat{\sigma_1}$	0.50	0.50	0.50	0.50	0.20	0.70
$\widehat{\sigma_2}$	0.50	0.50	0.50	0.50	0.70	0.20

Table 5.15: Values of the desired Delta $d = (d_1, d_2)$, the coefficients r_1, r_2, $\widehat{\sigma_1}, \widehat{\sigma_2}$ used in Experiments 7-12.

In tables 5.16-5.27, we can see less advantage for Armijo line search with respect to Back-tracking line search.

As shown in Figures 5.4, 5.5 and 5.6, for Exp. 7-12 we obtain similar results as in Exp. 1-6. However, the differences in the option prices are less pronounced, since the time-dependent interest rates and space-dependent volatilities are linearly varying between two extreme states.

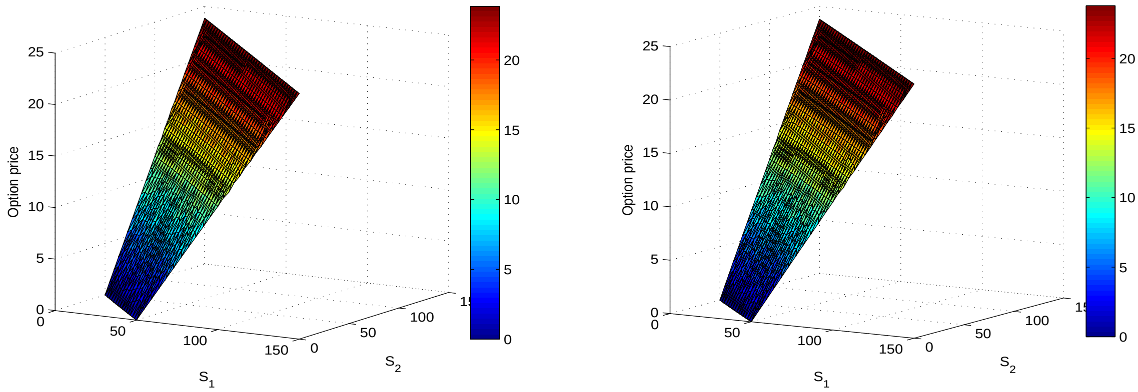

Figure 5.4: Option price at maturity for Exp. 7 (left) and Exp. 8 (right).

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	2.391e+02	1.862e+00	9	1.295e+02	5.762e-05

Table 5.16: Experiment 7 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	2.391e+02	1.862e+00	9	1.295e+02	6.479e-05

Table 5.17: Experiment 7 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	2.757e+02	2.481e+00	10	1.311e+02	1.647e-04
18	1.311e+02	8.946e-07			

Table 5.18: Experiment 8 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	2.757e+02	2.481e+00	10	1.311e+02	1.647e-04
17	1.311e+02	7.048e-06			

Table 5.19: Experiment 8 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

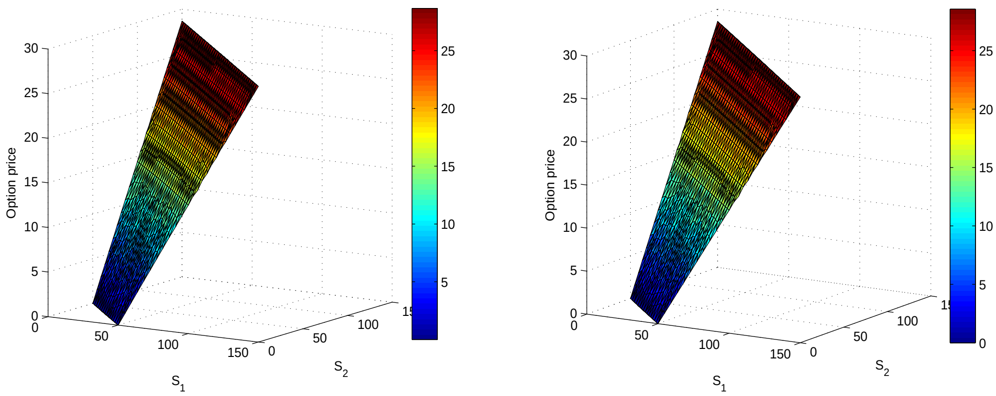

Figure 5.5: Option price at maturity for Exp. 9 (left) and Exp. 10 (right).

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	4.186e+02	2.773e-01	10	1.979e+02	3.935e-01
20	2.793e+01	3.203e-02	30	2.579e+01	3.236e-03
40	2.573e+01	3.599e-04	50	2.571e+01	1.040e-05
60	2.570e+01	3.055e-06	62	2.570e+01	6.694e-07

Table 5.20: Experiment 9 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	4.186e+02	2.773e-01	10	1.979e+02	3.935e-01
20	2.793e+01	3.203e-02	30	2.579e+01	3.236e-03
40	2.573e+01	3.599e-04	50	2.571e+01	1.040e-05
60	2.570e+01	3.055e-06	62	2.570e+01	6.694e-07

Table 5.21: Experiment 9 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	5.600e+02	2.245e+00	10	9.364e+01	4.049e-01
20	3.074e+01	1.574e-02	30	2.627e+01	4.690e-03
40	2.607e+01	1.043e-03	50	2.595e+01	3.049e-05
60	2.594e+01	6.592e-06	68	2.594e+01	7.526e-07

Table 5.22: Experiment 10 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	5.600e+02	2.245e+00	10	9.364e+01	4.049e-01
20	3.074e+01	1.574e-02	30	2.627e+01	4.690e-03
40	2.607e+01	1.043e-03	50	2.595e+01	3.049e-05
60	2.594e+01	6.592e-06	68	2.594e+01	7.526e-07

Table 5.23: Experiment 10 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

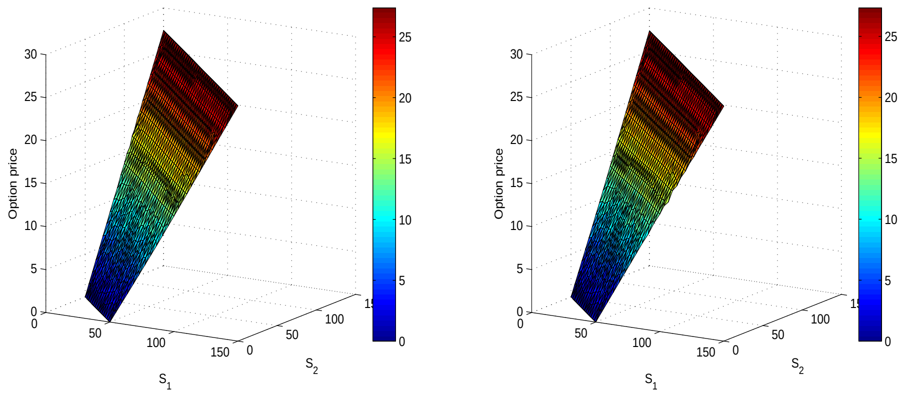

Figure 5.6: Option price at maturity for Exp. 11 (left) and Exp. 12 (right).

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	4.578e+02	5.722e-01	10	4.570e+01	2.132e-01
20	3.064e+01	1.264e-02	30	2.966e+01	3.134e-03
40	2.957e+01	5.495e-04	50	2.953e+01	3.030e-05
60	2.952e+01	8.243e-06	68	2.952e+01	9.405e-07

Table 5.24: Experiment 11 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	4.578e+02	5.722e-01	10	4.570e+01	2.132e-01
20	3.064e+01	1.264e-02	30	2.966e+01	3.134e-03
40	2.957e+01	5.495e-04	50	2.953e+01	3.030e-05
60	2.952e+01	8.243e-06	68	2.952e+01	9.405e-07

Table 5.25: Experiment 11 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	8.052e+01	7.929e-01	10	3.303e+01	6.941e-02
20	2.812e+01	2.219e-03	30	2.776e+01	3.760e-04
40	2.773e+01	9.660e-05	50	2.772e+01	7.333e-06
60	2.772e+01	1.313e-06	61	2.772e+01	8.437e-07

Table 5.26: Experiment 12 using Armijo line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$	ℓ	$J_{red}(u^{(\ell)})$	$\|\nabla J_{red}(u^{(\ell)})\|$
1	8.052e+01	7.929e-01	10	3.303e+01	6.941e-02
20	2.812e+01	2.219e-03	30	2.776e+01	3.760e-04
40	2.773e+01	9.660e-05	50	2.772e+01	7.333e-06
60	2.772e+01	1.313e-06	61	2.772e+01	8.437e-07

Table 5.27: Experiment 12 using Backtracking line search(Convergence history (maturity $t = T$)): Number ℓ of projected gradient iteration, value $J_{red}(u^{(\ell)})$ of the objective functional, and norm $\|\nabla J_{red}(u^{(\ell)})\|$ of the gradient.

Chapter 6

Conclusions

In this thesis, we have demonstrated that hedging with European Double Barrier Basket Options can be an attractive alternative to hedging with standard options or with futures contracts both for the buyer and for the seller. We have introduced a variant of such an option featuring multiple cash settlements that can be chosen in order to minimize a tracking-type objective functional in terms of the Delta of the option. Imposing bilateral constraints on the cash settlements, the problem can be formulated as a control constrained optimal control problem for the multidimensional Black-Scholes equation with Dirichlet boundary and final time control. The discretization in space by P1 conforming finite elements with respect to a simplicial triangulation of the spatial domain and in time by using the implicit Euler scheme with respect to a partition of the time interval leads to a finite dimensional constrained optimization problem which can be numerically solved by the projected gradient method with Armijo line search.

A reduction of the computational complexity could be achieved by projected model reduction based optimal control using, e.g., balanced truncation in case of time-independent data or Proper Orthogonal Decomposition (POD) in the general case. This will be the subject of future work.

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