FUNCTION THEORY ON THE QUANTUM ANNULUS AND OTHER DOMAINS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Meghna Mittal

August 2010

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Acknowledgements

"If I have seen further it is by standing on the shoulders of giants." — Isaac Newton, Letter to Robert Hooke, February 5, 1675.

It would be an understatement if I say I would like to extend my profound thanks to my thesis adviser, Prof. Vern Paulsen, for all his support and guidance throughout my thesis work. I have learned a lot from him, both on the research and non-technical side, and all that will help me in my future. I have great respect for his wide knowledge, logical way of thinking, and his ability to simplify complicated arguments. His patience and support helped me overcome many crisis situations and made this dissertation possible.

Next, I would like to thank Professor Scott McCullough, Professor Bernhard Bodmann, and Professor David Blecher for serving on my defense committee, and for their insightful suggestions for improving the presentations of this thesis.

I would like to extend sincere thanks to the Chairman of the Mathematics department at the University of Houston, Dr. Jeff Morgan for having faith in me. He has not only been a constant source of encouragement but also a great academic adviser. I would also like to thank the faculty and staff of the department.

I owe a lot to Dr. Dinesh Singh and other members of the Mathematical Sciences Foundation for giving me the opportunity to pursue my PhD here at the University of Houston. I sincerely appreciate their effort in building my strong math foundation and nurturing my research skills. Without their guidance, it would have been impossible to make it this far.

On a more personal note, I would like to thank my family (my father Suresh Mittal, my mother Suman Mittal, my brothers Varun and Abhinav, and my sister Ankita), to whom this work is dedicated, for their unparalleled care and love. I hope, with all my work throughout my PhD, I am able to make up for at least something for the time that I could not spend with them.

I am extremely fortunate to have my fiancé, Vivek Aseeja, who has given me nothing but unconditional love and support throughout the past years. I would like to thank Vivek for spending countless hours listening to me talk on about my research, while understanding very little of it over the phone and also for carefully reviewing the chapters of my thesis, politely pointing out glaring mistakes, and for expanding my vocabulary.

I would like to thank my dear friend and colleague, Sneh Lata with whom I worked closely during this thesis and my other friends both here in the Mathematics department over the years, and on "the outside" (you know who you are!).

I have thanked just a small fraction of people who have been instrumental for shaping my career so far and I ask forgiveness from those who have been omitted unintentionally.

Thank you all!

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Abstract

We are interested in studying a quantum analogue of the classical function theory on various domains in $\mathbb{C}^N$. The original motivation for this work comes from the work of Jim Agler which appeared in 1990 [2], and has origins in the work done by Nevanlinna and Pick in the area of classical interpolation theory. In the last two decades, the work of Agler has been generalized in multiple directions and for many domains, such as half planes by D. Kalyuzhnyi-Verbovetzkii in 2004 and the family of domains in $\mathbb{C}^N$ that are given by matrix-valued polynomials by Ambrozie-Timotin in 2003 and Ball-Bolotnikov in 2004.

In this thesis, we present a theory of special class of abelian operator algebras that we call *operator algebras of functions* which allows us to answer many interesting questions about these algebras in a unified manner. As a consequence, we are able to develop a quantized function theory for various domains that extends and unifies the work done by Agler, Ambrozie-Timotin, Ball-Bolotinov and D. Kalyuzhnyi-Verbovetzkii. We obtain analogous interpolation theorems, and prove that the algebras that we obtain are dual operator algebras. We also show that for many domains, supremums over all commuting tuples of operators satisfying certain inequalities are obtained over all commuting tuples of matrices. Also, we prove an abstract characterization of abelian operator algebras that are completely isometrically isomorphic to multiplier algebras of vector-valued reproducing kernel Hilbert spaces. Finally, we shall study a quantum analogue of annulus in great detail and present a study of some intrinsic properties of the algebra of functions defined on it.

Contents

Chapter 1

Background and Motivation

1.1 Nevanlinna-Pick Interpolation

There has been an intense interest in the classical Nevanlinna-Pick interpolation problem for purposes of many engineering applications such as system theory and H-infinity control theory. Attempts to extend this theory have lead to a great deal of development of various areas of mathematics such as operator theory, operator algebras, harmonic analysis, and complex function theory.

The *classical Nevanlinna-Pick interpolation* problem(NPP) was originally studied by Pick in 1916 [66] and independently by Nevanlinna in 1919 [57]. The statement of the problem is as follows. Given n points $z_1, \cdots, z_n$ in the open unit disk $\mathbb{D}$ and n points $w_1, \cdots, w_n$ in the open unit disk $\mathbb{D}$ characterize, in terms of the data $z_1, \cdots, z_n,\ w_1, \cdots, w_n$, the existence of a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_i) = w_i$. Pick's characterization was that such a function exists if and only if the matrix $\left(\frac{1-w_i\overline{w_j}}{1-z_i\overline{z_j}}\right)$ is positive definite. This matrix is referred to as the *Pick matrix*. We call a matrix A positive definite if for every

$x \in \mathbb{C}^N$ we have that $\langle Ax, x\rangle_{\mathbb{C}^N} \geq 0$. Pick's result can be restated as follows:

Theorem 1.1.1. *Given $2n$ points $z_1, \cdots, z_n$ and $w_1, \cdots, w_n$ in the open unit disk $\mathbb{D}$. Then there exists a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_i) = w_i$ if and only if there is a positive definite matrix (K_{ij}) such that*

$$1 - w_i\overline{w_j} = (1 - z_i\overline{z_j})K_{ij}$$

for every $1 \leq i,\ j \leq n$.

The original proof of this result by Pick relied on techniques from complex function theory. In particular, he used the Schwarz lemma and an inductive argument to obtain the result. Pick also established the fact that the solution to the interpolating function is unique if and only if the Pick matrix is singular. Nevanlinna working in Finland was unaware of Pick's result because of the First World War, though it was published in *Mathematische Annalen*. He also solved the same problem in [57]; however his conditions were rather implicit. His proof uses an idea of Schur [71], [72], and results in a different characterization. In 1929, Nevanlinna [58] gave a parametrization of all solutions in the nonunique case, i.e., when the Pick matrix is invertible. In fact, he characterized the set of all analytic function $f : \mathbb{D} \to \mathbb{D}$ in terms of positive definite functions. By a positive definite function, we mean a complex-valued function $K : X \times X \to \mathbb{C}$ such that for every finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq X$ we have that that matrix $(K(x_i, x_j))$ is positive definite.

Theorem 1.1.2. *A function $f : \mathbb{D} \to \mathbb{D}$ is analytic if and only if there exists a positive definite function $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ such that*

$$1 - f(z)\overline{f(w)} = (1 - z\overline{w})K(z, w) \quad \forall\ z,\ w \in \mathbb{D}. \tag{1.1}$$

We refer to this theorem as the *Nevanlinna Factorization theorem(NFT).*

Many people since Pick and Nevanlinna have contributed to the study of interpolation; in fact, too numerous for us to list here. The transparent nature of the statement of the problem has attracted researchers from various areas of analysis and hence it has been solved in many different ways. For example, in 1956, B.Sz.-Nagy and A. Koranyi [49] gave a proof of this result using Hilbert space techniques. In 1967, Sarason established the connection between the Nevanlinna-Pick problem and operator theory in his seminal paper [69]. His proof of Pick's theorem used the key idea that operators that commuted with the backward shift on an invariant subspace could be lifted to operators that commute with it on all of H^2 which was later generalized by B. Sz-Nagy and C. Foias in [41] to the commutant lifting theorem.

In order to apply operator theoretic techniques to this problem, Sarason did a reformulation of this problem. We will describe that reformulation here as this is of interest to us as well. The set of bounded analytic functions on the disk will be denoted by $H^\infty(\mathbb{D})$. The norm on $H^\infty(\mathbb{D})$ is the usual supremum norm

$$\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}.$$

When endowed with this norm, $H^\infty(\mathbb{D})$ is a Banach algebra and the maximum modulus theorem [8, page 134, Theorem 12] shows that the function $f : \mathbb{D} \to \mathbb{D}$ if and only if f is in the closed unit ball of $H^\infty(\mathbb{D})$. Therefore, the Nevanlinna-Pick theorem characterizes the existence of an element f in $H^\infty(\mathbb{D})$ such that $\|f\|_\infty \leq 1$ and $f(z_i) = w_i$.

Nevanlinna's factorization theorem completely characterizes the unit ball of $H^\infty(\mathbb{D})$ and can be restated as follows; f is in the closed unit ball of $H^\infty(\mathbb{D})$ if and only if there exists a positive definite function $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ such that

$$1 - f(z)\overline{f(w)} = (1 - z\overline{w})K(z, w)$$

for every $z,\ w \in \mathbb{D}$.

Many variants of Pick's theorem and Nevanlinna Factorization theorem are known but much remains unknown.

1. If one replaces the domain of the function (open unit disk) by some other domain in $\mathbb{C}^N$, then things get more complicated. For example, Abrahamse [1] gave a solution of the Pick's theorem for n-holed domains, but his conditions are nearly non-computable. Almost nothing is known for domains in several complex variable except the bidisk.

2. If one replaces the range of the function (open unit disk) by some other domain in $\mathbb{C}^N$, for instance, if we want a function that takes values in an annulus, or in the intersection of two disks, then the problem gets even harder.

In our work, we address the first of the above two variants of a Nevanlinna-Pick interpolation problem and Nevanlinna factorization theorem, but only for a special subclass of $H^\infty(G)$ where G is some "*nice*" domain in $\mathbb{C}^N$. We refer to these variants as the *Generalized Nevanlinna-Pick interpolation problem* (GNPP) and *Generalized Nevanlinna factorization theorem* (GNFT) respectively. We outline the motivational factor to study that subclass of $H^\infty(G)$ in the next few sections and the full description of it can be found in Chapter 3.

1.2 Von Neumann's Inequality

In 1951, von Neumann [81] proved that for every $f \in H^\infty(\mathbb{D})$, $\sup\{\|f(T)\|\} \leq \|f\|_\infty$, where the supremum is taken over all strict contractions $T \in B(\mathcal{H})$ and all Hilbert spaces $\mathcal{H}$, and $\|A\|$ is the operator norm of a bounded operator A on $\mathcal{H}$. This is referred to as von Neumann inequality. As an immediate consequence of this inequality, we find that for every $f \in H^\infty(\mathbb{D})$,

$$\|f\|_\infty = \sup\{\|f(T)\| : \|T\| < 1\}.$$

This remarkable result was a major contribution of von Neumann to an important field of functional analysis: Operator Theory.

Von Neumann proved this result by first proving that the inequality holds for the Mobius transformation of the disk, and then reducing the case of any general analytic function to this special case. Since then this result has been proved in many different ways. In the book [62] by Paulsen alone, there are five different proofs of this result. The most popular proof was given by Sz.-Nagy [76] in 1953 as an application of his dilation theorem. Sz.-Nagy's dilation theorem asserts that every contraction operator can be dilated to an unitary operator. In fact, it is known that Sz.-Nagy's dilation theorem [62, Theorem 4.3] is equivalent to the von Neumann inequality for polynomials.

A two variable analogue of Sz.-Nagy dilation theorem was proved by T. Ando [12] in 1963. The statement of Ando's dilation theorem is as follows:

Theorem 1.2.1. *Let T_1 and T_2 be commuting contractions on a Hilbert space $\mathcal{H}$. Then there exist a Hilbert space $\mathcal{K}$ that contains $\mathcal{H}$ as a subspace, and commuting unitaries U_1, U_2 on $\mathcal{K}$, such that*

$$T_1^n T_2^m = P_{\mathcal{H}} U_1^n U_2^m|_{\mathcal{H}}$$

for all non negative integers n, m.

It is also shown that this dilation theorem is equivalent to the two variable version of the von Neumann inequality for the matrices of polynomials which asserts that for every matrix of polynomials in two variables $P = (p_{ij})$, $\|P\|_\infty = \sup\{\|P(T_1, T_2)\| : \|T_i\| < 1\}$ where the supremum is taken over all commuting pairs of strict contractions $T_1,\ T_2 \in B(\mathcal{H})$

and all Hilbert spaces $\mathcal{H}$. As opposed to the one variable von Neumann inequality, there is only one proof known for this result that is by invoking Ando's dilation theorem which is proved using some geometric argument. Thus, the two variable version of the von Neumann inequality is referred as Ando's inequality.

It is surprising that the difference between the case of two and three or more contractions is still not very well understood. The corresponding analogue of Ando's theorem and the von Neumann inequality fails for three or more contractions. Several counterexamples have been produced but the first one was given by N. Th. Varopoulos [80] in 1974. Later in 1994, B.A. Lotto and T. Sterger[52] constructed three commuting diagonalizable contractions by perturbing Varopoulos's commuting contractions which also provides a counterexample to the multi-variable von Neumann inequality.

1.3 Agler Factorization

The remarkable extension of NPP and NFT for the bidisk were given by Jim Agler in 1988[2] and 1990[3], respectively. His statement of the Nevanlinna-Pick problem for the bidisk is as follows.

Theorem 1.3.1. *Given n points $z_1, \cdots, z_n$ in the open unit bidisk $\mathbb{D}^2$ and n points $w_1, \cdots, w_n$ in the open unit disk $\mathbb{D}$. Then there exists an analytic function $f : \mathbb{D}^2 \to \mathbb{D}$ such that $f(z_i) = w_i$ if and only if there exists positive definite matrices $K_i : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$, $1 \leq i \leq 2$, such that*

$$1 - w_i \overline{w_j} = (1 - z_i^1 \overline{z_j^1}) K_1(z_i, z_j) + (1 - z_i^2 \overline{z_j^2}) K_2(z_i, z_j) \tag{1.2}$$

for every $1 \leq i,\ j \leq n$, where $z_i = (z_i^1, z_i^2) \in \mathbb{D}^2$.

In a similar vein, Agler proved a natural extension of the Nevanlinna Factorization theorem for the bidisk.

Theorem 1.3.2. *A function* $f : \mathbb{D}^2 \to \mathbb{D}$ *is analytic if and only if there exist positive definite functions* $K_i : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C},\ 1 \le i \le 2,$ *such that for every* $z = (z_1, z_2),\ w = (w_1, w_2) \in \mathbb{D}^2$,

$$1 - f(z)\overline{f(w)} = (1 - z_1\overline{w_1})K_1(z,w) + (1 - z_2\overline{w_2})K_2(z,w). \tag{1.3}$$

An alternative proof of Pick's theorem using the notion of hyperconvex sets was given by Cole and Wermer [28]. Later in [60], Paulsen gave another proof of the same using an object that he called Schur Ideals which serves as a natural dual object for hyperconvex sets. The full matrix-valued version of this result was first obtained by Ball and Trent [19] and later independently by Agler and McCarthy [7]. Paulsen [61] also obtained the full matrix-valued version of this result in his follow up paper in which he defined the concept of "Matricial Schur Ideals". As for the Pick's theorem, several different proofs of the Nevanlinna Factorization theorem for the bidisk are known in the literature. But the key ingredient of all these proofs of NPP and NFT is Ando's inequality. In fact, Agler's Nevanlinna-Pick result (Pick's theorem for the bidisk) is known to be equivalent to Ando's inequality.

To explain Agler's idea of the proof, we need to first introduce the *Schur-Agler algebra* of analytic functions on a open unit polydisk $\mathbb{D}^N$. Given a natural number N and $I = (i_1, \dots, i_N) \in \mathbb{N}^N$ we set $z^I = z_1^{i_1} \cdots z_N^{i_N}$, so that every bounded analytic function $f : \mathbb{D}^N \to \mathbb{C}$ can be written as a power series, $f(z) = \sum_I a_I z^I$. If $T = (T_1, \dots, T_N)$ is an N-tuple of operators on some Hilbert space $\mathcal{H}$ which pairwise commute and satisfy $\|T_i\| < 1$ for every $i = 1, \dots, N$, then we will call T a *commuting N-tuple of strict contractions.* It is easily seen that if T is a commuting N-tuple of strict contractions then the power series $f(T) =$

$\sum_I a_I T^I$ converges and defines a bounded operator on $\mathcal{H}$. The space denoted by $H^\infty_{\mathcal{R}}(\mathbb{D}^N)$ is defined to be the set of analytic functions on $\mathbb{D}^N$ such that $\|f\|_{\mathcal{R}} = \sup\{\|f(T)\|\}$ is finite, where the supremum is taken over all sets of commuting N-tuples of strict contractions and all Hilbert spaces. In fact, the same supremum is attained by restricting to all commuting N-tuples of strict contractions on a fixed separable infinite dimensional Hilbert space. The significance of this subscript $\mathcal{R}$ will become clear in Chapter 3. It is fairly easy to see that $H^\infty_{\mathcal{R}}(\mathbb{D}^N)$ is a Banach algebra in the norm $\|\cdot\|_{\mathcal{R}}$. This algebra is called the Schur-Agler algebra and the set of all functions $f \in H^\infty_{\mathcal{R}}(\mathbb{D}^N)$ with $\|f\|_{\mathcal{R}} \leq 1$ is called the *Schur-Agler class* which is denoted by $\mathcal{SA}_N$. Later in [18], the notion of Schur-Agler class was extended to the operator-valued case and was denoted by $\mathcal{SA}_N(\mathfrak{E}, \mathfrak{E}')$ where $\mathfrak{E}, \mathfrak{E}'$ are Hilbert spaces.

$$\mathcal{SA}_N(\mathfrak{E}, \mathfrak{E}') = \{f : \mathbb{D}^N \to B(\mathfrak{E}, \mathfrak{E}') : \|f\|_{\mathcal{R}} \leq 1\}$$

Note that when the Hilbert space is one-dimensional, then every commuting N-tuple of strict contractions T is of the form $T = z = (z_1, \ldots, z_N) \in \mathbb{D}^N$, so that $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}^N\} \leq \|f\|_u$ and hence, $H^\infty_{\mathcal{R}}(\mathbb{D}^N) \subseteq H^\infty(\mathbb{D}^N)$, where this latter space denotes the set of bounded analytic functions on the polydisk $\mathbb{D}^N$. When $N = 1, 2$, it is known that these two spaces of functions are equal and that $\|.\|_{\mathcal{R}} = \|.\|_\infty$. For $N \geq 3$, it is known that these two norms are not equal, see Section 1.2. However, it is still unknown, for a general $N \geq 3$ if these two Banach spaces define the same sets of functions, since by the bounded inverse theorem, $H^\infty_{\mathcal{R}}(\mathbb{D}^N) = H^\infty(\mathbb{D}^N)$ if and only if there is a constant K_N such that $\|f\|_{\mathcal{R}} \leq K_N \|f\|_\infty$. The existence of such a constant is a problem that has been open since the early 1960's. For more details on all of these ideas one can see Chapters 5 and 18 of [62].

In [2] and [3], Jim Agler in fact proved the Pick's and the Nevanlinna factorization theorem for $\mathcal{SA}_N$. We now present the statement of the Nevanlinna factorization theorem in the form in which it appeared in [3].

Theorem 1.3.3. *A complex-valued function f is in the closed unit ball of $H^\infty_{\mathcal{R}}(\mathbb{D}^N)$ ($f \in \mathcal{SA}_N$) iff there exist positive definite functions $K_i, 1 \leq i \leq N$, such that*

$$1 - f(z)\overline{f(w)} = \sum_{i=1}^{N}(1 - z_i\overline{w_i})K_i(z, w) \tag{1.4}$$

for every $z = (z_1, \cdots, z_N), w = (w_1, \cdots, w_N) \in \mathbb{D}^N$.

This type of factorization is often referred to as **Agler Factorization**.

Other than the polydisk, there has been an extensive research done on the space of bounded holomorphic functions defined on the unit ball $\mathbb{B}_N$ in $\mathbb{C}^N$ with this new norm which is defined analogously as in the case of the polydisk. That is, the space denoted by $H^\infty_{\mathcal{R}}(\mathbb{B}^N)$ is defined to be the set of analytic functions on $\mathbb{B}^N$ such that $\|f\|_{\mathcal{R}} = \sup\{\|f(T)\|\}$ is finite, where the supremum is taken over all sets of commuting N-tuples of strict row contractions and all Hilbert spaces. By a row contraction, we mean a tuple of operators $(T_1, T_2, \cdots, T_N)$ that satisfy the condition $\sum_{i=1}^{N} T_iT_i^* < 1$. This was first studied by S.W. Drury[39] in the context of von Neumann's inequality. The set of all functions $f \in H^\infty_{\mathcal{R}}(\mathbb{B}^N)$ with $\|f\|_{\mathcal{R}} \leq 1$ is called the *Schur-Agler class for the unit ball.* Several others such as Davidson and Pitts [35], Popescu [68] and Agler and McCarthy [6] have worked on this space and have proved the scalar-valued Nevanlinna-Pick type result. Later this result was generalized for the matrix-valued functions which appeared in the work of Arveson [15] and Agler and McCarthy [7]. The algebra $H^\infty_{\mathcal{R}}(\mathbb{B}^N)$ is also sometimes referred as the Arveson-Drury-Popescu algebra.

This motivated researchers to study H^∞ spaces on different domains equipped with this new norm and consider these as the right object for the generalization of NPP and NFT. Since then, there has been a constant progress in this direction. Essentially our work is also centered around obtaining such results for these spaces. In the following section, we would like to mention the work done by Ambrozie-Timotin [10], Ball-Bolotnikov[17] since it is closely related to our work which we will describe in Chapter 3.

Before we move on to the next section, we would like to remark that even to this date very little is known about the classical H^∞ spaces even for the generic domains such as polydisk $\mathbb{D}^N$ and unit ball $\mathbb{B}^N$ for $N > 2$ in the context of NFT and NPP. No factorization result exists for $H^\infty(\mathbb{B}^N),\ N > 2$. In contrast, there was no factorization result known for $H^\infty(\mathbb{D}^N),\ N > 2$ until very recently. In 2009, A. Grinshpan, D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov and H. Woerdeman [43] gave a necessary condition to solve the GNPP for the polydisk. Still, it is unknown if their condition is also sufficient. They proved this result by obtaining a factorization that is analogous to Agler factorization(GNFT). We state their factorization result in the scalar-valued case but it holds in the operator-valued case as well.

Theorem 1.3.4. *[43] A necessary condition for any complex-valued function f to be in the closed unit ball of $H^\infty(\mathbb{D}^N)$, is that for every $1 \leq p < q \leq N$, there are positive semi-definite matrices K^p and K^q such that*

$$1 - f(z)f(w)^* = \prod_{i \neq p}(1 - z_i\overline{w_i})K^p(z,w) + \prod_{j \neq q}(1 - z_j\overline{w_j})K^q(z,w).$$

1.4 Ball-Bolotnikov Factorization

Inspired by the work of Agler, Ambrozie, and Timotin [10] defined a *generalized* Schur-Agler class of functions on some *natural class* of domains in $\mathbb{C}^N$ and gave a unified proof of the existing Nevanlinna-Pick type result for domains such as polydisk, unit ball and for other domains in this *natural class*.

Let us keep the notation in mind that $M_{p,q}$ denote the set of $p \times q$ matrices over $\mathbb{C}$ and in particular when $p = q$, then M_p denote the set of $p \times p$ matrices over $\mathbb{C}$. Ambrozie-Timotin considered the following class of domains which are defined by a multivariable matrix-valued polynomial, $P : \mathbb{C}^N \to M_{p,q}(\mathbb{C})$,

$$\Omega = \{z \in \mathbb{C}^N : \|P(z)\| < 1\}.$$

It is easy to see that the polydisk and the unit ball are examples of the domain which are defined using polynomials. Indeed if we take $P(z) = diag(z_1, z_2, \cdots, z_N) \in M_N$ then $\Omega = \mathbb{D}^N$ and if we take $P(z) = (z_1, z_2, \cdots, z_N) \in C^N$ then $\Omega = \mathbb{B}_N$.

Their idea was to study a space of analytic functions that generalizes the Schur-Agler class, that is, the space of analytic functions that satisfies the von Neumann inequality. To be able to understand their approach, we need the following. Let f be an analytic function defined on an open set $G \subseteq C^N$ and $T = (T_1, \cdots, T_N)$ be a pairwise commuting N-tuple of operator in $B(\mathcal{H})$. To be able to make sense of $f(T)$, we need a functional calculus. We would like something which is analogous to the Riesz-Dunford functional calculus for a single operator [29], whereby one can define $f(T)$ for every function analytic in a neighbourood of the spectrum of T. Moreover, we would like this spectrum to be as small as possible so that the functional calculus is as large as possible. There are many ways one can define the spectrum of an N-tuple of operators. The best way, in the sense of the above desired properties, seems to be the *Taylor spectrum* which was introduced by J.L. Taylor in [77] and [78]. We use the notation $\sigma(T)$ for the Taylor spectrum of the operator T and is defined as the set of all points $\lambda \in \mathbb{C}^N$ so that the *Koszul complex* of $T - \lambda I$ is not *exact*. For the precise meaning of the terms used in the definition of the Taylor spectrum, we refer the reader to [77], [78], [79]. Here, we shall list some of its important properties that we will be using throughout this thesis:

1. The Taylor spectrum $\sigma(T)$ is compact and non-empty.

2. If $p : \mathbb{C}^N \to \mathbb{C}^M$ is a polynomial mapping, then

$$\sigma(p(T)) = p(\sigma(T)).$$

3. Let G be an open set in $\mathbb{C}^N$ containing $\sigma(T)$. Then there is a continuous unital homomorphism $\pi : Hol(G) \to B(\mathcal{H})$ from the algebra of holomorphic functions on G into $B(\mathcal{H})$ which is defined via the map $\pi(f) = f(T)$.

4. For all bounded open sets G_1 for which $\sigma(T) \subseteq G_1 \subseteq G_1{}^- \subseteq G$, there exists a constant C (depending on T and G_1) such that

$$\|f(T)\| \leq C \sup\{|f(z)| : z \in G_1\}.$$

We refer the reader to [11] for short proofs of some of the above properties and to [32] for a detailed exposition on the Taylor spectrum.

Ambrozie-Timotin proved that every commuting N-tuple of operator T that satisfies $\|P(T)\| < 1$, the Taylor spectrum of T is contained in the domain $\Omega = \{z \in \mathbb{C}^N : \|P(z)\| < 1\}$. Thus, by using the Taylor functional calculus, $f(T)$ can be defined for every f analytic on ω.. We are now in a position to define the generalized Schur-Agler class of analytic functions,

$$\mathcal{SA}_P = \{f : \Omega \to \mathbb{C} : \ \|f(T)\| \leq 1 \ \ whenever \ \ \|P(T)\| < 1\}.$$

Their main result generalizes both Nevanlinna Factorization theorem and the Nevanlinna-Pick result for the above defined generalized Schur-Agler class. The statement of their result is as follows:

Theorem 1.4.1. *Given a subset $X \subseteq \Omega$ and a complex-valued function $\phi : X \to \mathbb{C}$. Then there is a function $\Phi \in \mathcal{SA}_P$ such that $\Phi|_X = \phi$ iff there exist a matrix-valued positive* *definite function $\Gamma : X \times X \to M_p$ such that*

$$1 - \phi(z)\phi(w)^* = Tr((I - P(z)P(w)^*)\Gamma(z,w)).$$

Remark 1.4.2. *Note that if we take $X = \Omega$ then this theorem gives a GNFT and if we take X to be finite then this gives a solution of the GNPP for the above defined generalized Schur-Agler class. Also, it is easy to see that this result unifies the results for the two generic settings (Polydisk and Unit Ball) defined above and covers some more interesting examples.*

In 2004, Ball and Bolotnikov [17] extended this work of Ambrozie-Timotin [10] to the operator-valued case. They defined a generalized Schur-Agler class as the class of operator-valued analytic functions that satisfies the von Neumann inequality.

$$\mathcal{SA}_P(\mathfrak{E}, \mathfrak{E}') = \{f : \Omega \to B(\mathfrak{E}, \mathfrak{E}') : \; \|f(T)\| \leq 1 \;\; \text{w}\textit{henever} \;\; \|P(T)\| < 1\}.$$

In particular, if we take $\mathfrak{E} = \mathfrak{E}' = \mathbb{C}$ then $\mathcal{SA}_P(\mathfrak{E}, \mathfrak{E}')$ coincides with the class introduced by Ambrozie-Timotin. Their main result also proves both NFT and NPP but for their operator-valued generalized Schur-Agler class. However, no new factorization result for the class introduced by Ambrozie-Timotin arise in this way: the factorization obtained by Ball-Bolotnikov coincides with the existing one.

Their statement of the main result had many equivalences but here we only state the ones that are relevant to our discussion. Also, the factorization that we state here is somewhat different looking, though equivalent to the one that was stated by Ball-Bolotnikobv in [17] as part of their main result. This formulation of the factorization makes it easy for us to be able to compare their result with the factorization obtained by Ambrozie-Timotin.

Theorem 1.4.3. *Let Ω and P be as defined above. Given a subset $X \subseteq \Omega$ and a operator-valued function $\phi : X \to B(\mathfrak{E}, \mathfrak{E}')$. Then there is a function $\Phi \in \mathcal{SA}_P(\mathfrak{E}, \mathfrak{E}')$ such that $\Phi|_X = \phi$ iff there exist a positive definite kernel $K : X \times X \to B(\mathbb{C}^p \otimes \mathfrak{E}')$ such that*

$$I_{\mathfrak{E}'} - \phi(z)\phi(w)^* = Tr\left((I - P(z)P(w)^*)K(z, w)\right).$$

Now, we would like to obtain the factorization that they state in their main result as it will be convenient for the reader to compare this factorization with the one that we obtain in Chapter 3. Before we assert this, we would like to record a useful fact about positive definite kernels.

Theorem 1.4.4. *Let K be a $B(\mathcal{L})-$valued positive definite kernel on some set X. Then there is a Hilbert space $\mathcal{L}$ and functions $F : X \to B(\mathcal{H}, \mathcal{L})$ such that K can be represented as $K(z, w) = F(z)F(w)^*$.*

For the proof of the above result, we refer the reader to [5, Theorem 2.62].

Note that the positive definite kernel obtained in 1.4.3 can be written as $\Gamma(z, w) = G(z)G(w)^*$ where $G(z) \in B(\mathcal{H}, \mathbb{C}^p \otimes \mathfrak{E}')$ for some Hilbert space $\mathcal{H}$. Further, we can write

$$G(z) = \begin{bmatrix} G_1(z) \\ \vdots \\ G_p(z) \end{bmatrix}$$

where $G_i(z) \in B(\mathcal{H}, \mathfrak{E}')$ for every $1 \leq i \leq p$. The direct calculations yield that the factorization stated in 1.4.3 is equivalent to the following factorization. For details, please see [17, Page 54].

$$I_{\mathfrak{E}'} - \phi(z)\phi(w)^* = H(z)(I_{\mathbb{C}^p \otimes \mathfrak{E}'} - P(z)P(w)^*)H(w)^*$$

where $H(z) = [G_1(z), \cdots, G_p(z)]$ is a function defined on $B(\mathbb{C}^p \otimes \mathcal{H}, \mathfrak{E}')$ for some Hilbert space $\mathcal{H}$. We refer to this factorization as **Agler-Ball-Bolotnikov Factorization**.

The approach to the GNFT and GNPP used by Agler, Ambrozie-Timotin and Ball-Bolotnikov is quite similar. Their main tools include separation arguments from Banach space theory and methods from the dilation theory. We employ methods from the theory of operator algebras to extend the work of Ball and Bolotnikov with slight but necessary modification which offers a new look at the Generalized Schur-Agler class. In a true sense, our work is an extension of the work of Ambrozie-Timotin, this will become clear in later sections.

We summarize the events described in the earlier sections and the aim of this thesis using the following diagram. Let P be the matrix-valued polynomial and F be the matrix-valued analytic function.

$$\underbrace{\text{GNFT for } SA_N}_{\text{Agler}} \longrightarrow \underbrace{\text{GNFT for} SA_P}_{\text{Ambrozie-Timotin}} \longrightarrow \underbrace{\text{GNFT for} SA_P(\mathfrak{E}, \mathfrak{E}')}_{\text{Ball-Bolotnikov}} \longrightarrow \underbrace{\text{GNFT for } SA_F(\mathfrak{E}, \mathfrak{E}')}_{\text{Thesis}}.$$

1.5 Overview of Thesis

The main goal of this thesis is to study the generalized Schur-Agler class of functions defined on the "most" general class of domains using methods from the theory of operator algebras and also to reformulate these ideas in terms of algebras of operators. In the following chapter, we develop a theory of a "special" subclass of operator algebras that we call *Operator Algebras of Functions.* These objects are a nice blend of objects like operator algebras, function algebras, and reproducing kernel Hilbert spaces. Thus, it is not very surprising to find out that these objects possess some interesting theory.

In Chapter 3, we give a formal terminology to the process that has been carried out by Agler, Ambrozie-Timotin, and Ball-Bolotnikov and is described above. Basically, we formalize the process by which one begins with a complex domain defined by a family of inequalities and creates a quantized version of the domain by considering the operators that satisfy the same inequalities and then studies the function theory on the quantized domain. Furthermore, as an application of the theory of *Operator Algebras of Functions*, we prove a number of new facts about the algebras of bounded analytic functions on these quantized domains. We prove that they are dual operator algebras, that they can be represented as the multiplier algebras of reproducing kernel Hilbert spaces and that appropriate analogues of Agler-Ball-Bolotnikov factorization theorem hold. We also prove that in many cases it is sufficient to replace the operator variables by matrices when defining the norms.

In Chapter 4, we show that the existence of Fejér-like kernels gives us another way to prove GNFT for a generalized Schur-Agler class. In a joint work with Lata and Paulsen [51], we gave a shorter and more informative proof of the Agler result for the polydisk using the existence of these kernels. In Section 4.2, we show that Fejér-like kernels exist for many domains in $\mathbb{C}^N$ such as annulus and unit ball in $\mathbb{C}^N$ for any norm. This allows us to extend the ideas of the proof of the Agler's result for the polydisk to the case of the annulus and the unit balls in $\mathbb{C}^N$ for some norm.

Finally in Chapter 5, we present the case study of the function theory on the space of bounded analytic functions on *quantum annulus*. By using a natural embedding into the bidisk, we present a third proof of the GNFT and an expected solution of the GNPP for this particular domain. In Section 5.3, we introduce the generalized notion of pseudo-hyperbolic distance induced by an operator algebra of functions on any set X and establish a connection of this distance formula with the two dimensional representations of operator algebra of functions. In particular, for the quantum annulus, we prove a direct generalization of Schwarz-Pick lemma.

In Section 5.4, we mention two different approaches to estimate the constant K that occurs in the inequality $\|f\|_R \leq K\|f\|_\infty$. One uses the pseudohyperbolic distance and the other one uses the idea of hyperconvex sets [27], [60].

The results in Chapter 2 and 3 are joint work with my adviser, Vern Paulsen and appears in [55]. The parts of this thesis which do not appear in [55], are also done under the constant guidance of my adviser.

Chapter 2

Operator Algebras of Functions

2.1 Introduction

Operator algebras originated in quantum mechanics, where operators were used to represent physical quantities and describe noncommutative phenomena found in nature. Today operator algebras have found widespread application to such diverse areas as group representations, dynamical systems, differential geometry, knot theory, and various areas of physics.

A *concrete operator algebra* $\mathcal{A}$ is just a subalgebra of $B(\mathcal{H})$, the bounded operators on a Hilbert space $\mathcal{H}$. The operator norm on $B(\mathcal{H})$ gives rise to a norm on $\mathcal{A}$. Moreover, the identification

$$M_n(\mathcal{A}) \cong \mathcal{A} \otimes M_n \subseteq B(\mathcal{H} \otimes \mathbb{C}^n) \cong B(\mathcal{H}^n) = B(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \ copies})$$

endows the matrices over $\mathcal{A}$ with a family of norms in a natural way, where M_n denotes the algebra of $n \times n$ matrices. The collection of these norms $\{\|.\|_n\}$ is called the matrix norm structure of $\mathcal{A}$.

Given two operator algebras $\mathcal{A}$ and $\mathcal{B}$ and a map $\phi : \mathcal{A} \to \mathcal{B}$, we obtain maps $\phi^{(n)}$: $M_n(\mathcal{A}) \to M_n(\mathcal{B})$ via the formula

$$\phi^{(n)}((a_{i,j})) = (\phi(a_{i,j})).$$

This map $\phi^{(n)}$ is called the n^{th}-amplification of ϕ. It is natural to consider such maps between operator algebras because of their matrix norm structure. Before the arrival of the theory of operator algebras, such amplifications were extensively studied for C^*-algebras. We say ϕ is completely bounded if each ϕ_n is bounded and $\|\phi\|_{cb} := \sup_n \|\phi^{(n)}\| < \infty$. We say ϕ is a complete contraction if $\|\phi\|_{cb} \leq 1$ and a complete isometry if each $\phi^{(n)}$ is an isometry. In particular, if $\phi^{(n)}$ is a contraction, then we say ϕ is n-contractive and if $\phi^{(n)}$ is an isometry, then we say ϕ is an n-isometry.

It is common practice to identify two operator algebras $\mathcal{A}$ and $\mathcal{B}$ as being the "same" if and only if there exists an algebra isomorphism $\pi : \mathcal{A} \to \mathcal{B}$ that is not only an isometry, but which also preserves all the matrix norms, that is such that $\|(\pi(a_{i,j}))\|_{M_n(\mathcal{B})} = \|(a_{i,j})\|_{M_n(\mathcal{A})}$, for every n and every element $(a_{i,j}) \in M_n(\mathcal{A})$. Such a map π is called a *completely isometric isomorphism.*

An algebra $\mathcal{A}$ with matrix norms $\{\|.\|_n\}$ is called an *abstract operator algebra* if it satisfies the following axioms that are called *Blecher-Ruan-Sinclair axioms*, abbreviated as BRS axioms:

(1) $\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_n\|\beta\|$, for all $n \in \mathbb{N}$ and all $\alpha,\ \beta \in M_n$, and $x \in M_n(\mathcal{A})$.

(2) $\|x \oplus y\|_{m+n} = max\{\|x\|_n, \|y\|_m\}$ for all $x \in M_n(\mathcal{A})$ and $y \in M_m(\mathcal{A})$. Here $\oplus$ denotes the diagonal direct sum of matrices.

(3) $\|xy\|_n \leq \|x\|_n\|y\|_n$ for all $x,\ y \in M_n(\mathcal{A})$.

Axiom (1) and (2) together are called Ruan's axioms (characterizes an operator space) and when (3) hold true then we say that the product on the algebra $\mathcal{A}$ is completely contractive. If $\mathcal{A}$ also has a unit e with $\|e\| = 1$, then we call $\mathcal{A}$ an *abstract unital operator algebra*. For the purpose of our work in this thesis, we may assume that our operator algebras are unital.

In 1990, Blecher, Ruan, and Sinclair [25] gave their abstract characterizations of operator algebras that "frees us" from always having to regard operator algebras as concrete subalgebras of some Hilbert space and at the same time, allows us to consider them as concrete whenever needed. This characterization result serves as the fundamental result in the theory of operator algebras and since then its theory has greatly evolved. The following theorem, known as the BRS theorem, shows that every abstract unital operator algebra is, in fact, a concrete operate algebra.

Theorem 2.1.1. *Let $\mathcal{A}$ be an* unital abstract operator algebra*. Then there exist a Hilbert space $\mathcal{H}$ and a completely isometric homomorphism $\phi : \mathcal{A} \to B(\mathcal{H})$.*

For more details on the abstract theory of operator algebras, see [23], [62] or [67].

In this chapter we present a theory for a special class of abstract abelian operator algebras that we call *operator algebras of functions*. There are a number of significant reasons for us to develop the theory of such algebras. These algebras contains many important examples arising in function theoretic operator theory, including the Schur-Agler and the Arveson-Drury-Popescu algebras. In the next chapter, we will exhibit the application of the theory of these operator algebras to study "quantized function theories" on various domains.

In addition to this impressive application, this subclass of operator algebras seems to be interesting in its own right. The work that follows will show that this theory allows us to answer certain kinds of questions about such algebras in a unified manner. We will prove an abstract characterization of abelian operator algebras that are completely isometrically isomorphic to multiplier algebras of vector-valued reproducing kernel Hilbert spaces. This result can be viewed as expanding on the Agler-McCarthy concept of *realizable algebras* [5].

Our results will show that under certain mild hypotheses, operator algebra norms, which are defined by taking the supremum of certain families of operators on Hilbert spaces of arbitrary dimensions, can be obtained by restricting the family of operators to finite dimensional Hilbert spaces. Thus, in a certain sense, which will be explained later, our results give conditions that guarantee that an algebra is *residually finite dimensional.*

This chapter is organized as follows. In Section 2.2, we introduce a subclass of operator algebras of functions: local and BPW (stands for bounded pointwise) complete operator algebras of functions. To illustrate and justify the natural definitions of these properties, we dedicate Section 2.2.3 for the examples of these algebras. In the subsequent section, we show that this subclass of operator algebra of functions can be characterized as the class of multiplier algebra of a vector-valued reproducing kernel Hilbert space. In the same section, we provide sufficient condition for an operator algebra of function to be a dual operator algebra. Finally, in the last section of this chapter, we extend the notion of RFD from C^*-algebras to operator algebras. In closing, we illustrate the connection of the theory developed in the earlier sections with the residually finite dimensional operator algebras of functions through some results and examples.

We now give the relevant definitions. Recall that given any set X the set of all complex-valued functions on X is an algebra over the field of complex numbers.

Definition 2.1.2. *We call $\mathcal{A}$ an* **operator algebra of functions***(abbreviated as OPAF) on a set X provided:*

1. *$\mathcal{A}$ is a subalgebra of the algebra of functions on X equipped with the usual point-wise multiplication,*

2. *$\mathcal{A}$ separates the points of X and contains the constant functions,*

3. *for each n, $M_n(\mathcal{A})$ is equipped with a norm $\|.\|_{M_n(\mathcal{A})}$, such that the set of norms satisfy the BRS axioms [25] to be an abstract operator algebra,*

4. *for each $x \in X$, the evaluation functional, $\pi_x : \mathcal{A} \to \mathbb{C}$, given by $\pi_x(f) = f(x)$ is bounded.*

A few remarks and observations are in order. First note that if $\mathcal{A}$ is an operator algebra of functions on X and $\mathcal{B} \subseteq \mathcal{A}$ is any subalgebra, which contains the constant functions and still separates points, then $\mathcal{B}$, equipped with the norms that $M_n(\mathcal{B})$ inherits as a subspace of $M_n(\mathcal{A})$ is still an operator algebra of functions.

The basic example of an operator algebra of functions is $\ell^\infty(X)$, the algebra of all bounded functions on X. If for $(f_{i,j}) \in M_n(\ell^\infty(X))$ we set

$$\|(f_{i,j})\|_{M_n(\ell^\infty(X))} = \|(f_{i,j})\|_\infty \equiv \sup\{\|(f_{i,j}(x))\|_{M_n} : x \in X\},$$

where $\|\cdot\|_{M_n}$ is the norm on M_n obtained via the identification $M_n = B(\mathbb{C}^n)$, then it readily follows that properties (1)–(4) of the above definition are satisfied. Thus, $\ell^\infty(X)$ is an operator algebra of functions on X in our sense and any subalgebra of $\ell^\infty(X)$ that contains the constants and separates points will be an operator algebra of functions on X when equipped with the subspace norms.

Proposition 2.1.3. *Let $\mathcal{A}$ be an operator algebra of functions on X, then $\mathcal{A} \subseteq \ell^\infty(X)$, and for every n and every $(f_{i,j}) \in M_n(\mathcal{A})$, we have $\|(f_{i,j})\|_\infty \leq \|(f_{i,j})\|_{M_n(\mathcal{A})}$.*

Proof. Since $\pi_x : \mathcal{A} \to \mathbb{C}$ is bounded and the norm is sub-multiplicative, we have that for any $f \in \mathcal{A}$, $|f(x)|^n = |\pi_x(f^n)| \leq \|\pi_x\|\|f^n\| \leq \|\pi_x\|\|f\|^n$. Taking the n-th root of each side of this inequality and letting $n \to +\infty$, yields $|f(x)| \leq \|f\|$, and hence, $f \in \ell^\infty(X)$. Note also that $\|\pi_x\| = 1$.

We repeat this argument for the amplification of π_x. Since every bounded, linear functional on an operator space is completely bounded and the norm and the cb-norm are equal, we have that $\|\pi_x\|_{cb} = \|\pi_x\| = 1$. Thus, for $(f_{i,j}) \in M_n(\mathcal{A})$, we have $\|(f_{i,j}(x))\|_{M_n} = \|(\pi_x(f_{i,j}))\|_{M_n} \leq \|\pi_x\|_{cb}\|f_{i,j}\| \leq \|(f_{i,j})\|_{M_n(\mathcal{A})}$. □

2.2 Local and BPW Complete OPAF

We have divided this section into three subsections. In the first two, we introduce the concept of the local and BPW complete operator algebra of functions and in the third subsection, we give examples to illustrate the concept.

2.2.1 Local OPAF

Given an operator algebra $\mathcal{A}$ of functions on a set X and $F = \{x_1, ..., x_k\}$ a set of $k \geq 1$ distinct points in X, we set $I_F = \{f \in \mathcal{A} : f(x) = 0 \text{ for all } x \in F\}$. Note that for each n, $M_n(I_F) = \{f \in M_n(\mathcal{A}) : f(x) = 0 \text{ for all } x \in F\}$. The quotient space $\mathcal{A}/I_F$ has a natural set of matrix norms given by defining $\|(f_{i,j} + I_F)\| = \inf\{\|(f_{i,j} + g_{i,j})\|_{M_n(\mathcal{A})} : g_{i,j} \in I_F\}$. Alternatively, this is the norm on $M_n(\mathcal{A}/I_F)$ that comes via the identification, $M_n(\mathcal{A}/I_F) = M_n(\mathcal{A})/M_n(I_F)$, where the latter space is given its quotient norm. It is easily checked that this family of matrix norms satisfies the BRS axioms and so gives $\mathcal{A}/I_F$ the structure of an abstract operator algebra as in [25], the quotient of any operator algebra is an operator algebra.

We let $\pi_F(f) = f + I_F$ denote the quotient map $\pi_F : \mathcal{A} \to \mathcal{A}/I_F$ so that for each $n, \pi_F^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{A}/I_F) \cong M_n(\mathcal{A})/M_n(I_F)$.

Since $\mathcal{A}$ is an algebra which separates points on X and contains constant functions, it follows that there exist functions $f_1, ..., f_k \in \mathcal{A}$, such that $f_i(x_j) = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker's delta symbol. If we set $E_j = \pi_F(f_j)$, then it is easily seen that whenever $f \in \mathcal{A}$ and $f(x_i) = \lambda_i, i = 1, ..., k$, then $\pi_F(f) = \lambda_1 E_1 + \cdots + \lambda_k E_k$. Moreover, $E_i E_j = \delta_{i,j} E_i$, and $E_1 + \cdots + E_k = 1$, where 1 denotes the identity of the algebra $\mathcal{A}/I_F$. Thus, $\mathcal{A}/I_F = \text{span}\{E_1, ..., E_k\}$, is a unital algebra spanned by k commuting idempotents. Such algebras were called *k-idempotent operator algebras* in [61] and we will use a number of results from that paper.

Definition 2.2.1. *An operator algebra of functions $\mathcal{A}$ on a set X, is called a* **local operator algebra of functions** *if it satisfies*

$$\sup_F \|\pi_F^{(n)}((f_{i,j}))\| = \|(f_{i,j})\| \text{ for all } (f_{i,j}) \in M_n(\mathcal{A}) \text{ and for every } n,$$

where the supremum is taken over all finite subsets $F \subseteq X$.

The following result shows that every operator algebra of functions can be re-normed so that it becomes local. Since there exist more than one norm structure on an algebra, we use a simple notation $A \subseteq_{cc} B$ to indicate that $A \subseteq B$ completely contractively, and

$A \subseteq_{ci} B$ if $A \subseteq B$ completely isometrically. Whenever A and B are the same as sets but have different norm structure then we write $A \subsetneq_{cc} B$ if the identity map from A to B is completely contractive but not a complete isometry.

Proposition 2.2.2. *Let $\mathcal{A}$ be an operator algebra of functions on X, let $\mathcal{A}_L = \mathcal{A}$ and define a family of matrix norms on $\mathcal{A}_L$, by setting $\|(f_{i,j})\|_{M_n(\mathcal{A}_L)} = \sup_F \|(\pi_F(f_{i,j}))\|_{M_n(\mathcal{A}/I_F)}$, where the supremum is taken over all finite subsets of X. Then $\mathcal{A}_L$ is a local operator algebra of functions on X and the identity map, $id : \mathcal{A} \to \mathcal{A}_L$, is completely contractive.*

Proof. It is clear from the definition of the norms on $\mathcal{A}_L$ that the identity map is completely contractive and it is readily checked that $\mathcal{A}_L$ is an operator algebra of functions on X.

Let $\tilde{\pi}_F : \mathcal{A}_L \to \mathcal{A}_L/I_F$, denote the quotient map, so that $\|\tilde{\pi}_F(f)\| = \inf\{\|f+g\|_{\mathcal{A}_L} : g \in I_F\} \leq \inf\{\|f+g\|_{\mathcal{A}} : g \in I_F\} = \|\pi_F(f)\|$, since $\|f+g\|_{A_L} \leq \|f+g\|_{\mathcal{A}}$. We claim that for any $f \in \mathcal{A}$, and any finite subset $F \subseteq X$, we have that $\|\pi_F(f)\| = \|\tilde{\pi}_F(f)\|$. To see the other inequality note that for $g \in I_F$, and $G \subseteq X$ a finite set, we have $\|f+g\|_L = \sup_G \|\pi_G(f+g)\| \geq \|\pi_F(f+g)\| = \|\pi_F(f)\|$. Hence, $\|\tilde{\pi}_F(f)\| \geq \|\pi_F(f)\|$, and equality follows. A similar calculation shows that $\|(\tilde{\pi}_F(f_{i,j}))\| = \|(\pi_F(f_{i,j}))\|$, for any matrix of functions. Now it easily follows that $\mathcal{A}_L$ is local, since

$$\sup_F \|(\tilde{\pi}_F(f_{i,j}))\| = \sup_F \|(\pi_F(f_{i,j}))\| = \|(f_{i,j})\|_{M_n(\mathcal{A}_L)}.$$

□

2.2.2 BPW Complete OPAF

In this section, we introduce a notion of BPW complete operator algebra of functions which seem to connect with the theory of local operator algebra of functions defined in the earlier section quite naturally.

Definition 2.2.3. *Given an operator algebra of functions $\mathcal{A}$ on X we say that $f : X \to \mathbb{C}$ is a* **BPW limit** *of $\mathcal{A}$ if there exists a uniformly bounded net $(f_\lambda)_\lambda \in \mathcal{A}$ that converges pointwise on X to f. We let $\tilde{\mathcal{A}}$ denote the set of BPW limits of functions in $\mathcal{A}$. We say that $\mathcal{A}$ is* **BPW complete,** *if $\mathcal{A} = \tilde{\mathcal{A}}$.*

Given $(f_{i,j}) \in M_n(\tilde{\mathcal{A}})$, we set

$$\|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} = \inf\{C : (f_{i,j}(x)) = \lim_\lambda (f^\lambda_{i,j}(x)) \text{ and } (f^\lambda_{i,j}) \in M_n(\mathcal{A}) \text{ with } \|(f^\lambda_{i,j})\| \leq C\}.$$

It is easily checked that for each n, the above formula defines a norm on $M_n(\tilde{\mathcal{A}})$. It is also easily checked that a matrix-valued function, $(f_{i,j}) : X \to M_n$ is the pointwise limit of a uniformly bounded net $(f^\lambda_{i,j}) \in M_n(\mathcal{A})$ if and only if $f_{i,j} \in \tilde{\mathcal{A}}$ for every i, j.

Lemma 2.2.4. *Let $\mathcal{A}$ be an operator algebra of functions on X and let $(f_{i,j}) \in M_n(\tilde{\mathcal{A}})$. Then*

$$\|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} = \inf\{C : \text{for each finite } F \subseteq X \text{ there exists } g^F_{i,j} \in \mathcal{A} \text{ with } (f_{i,j}\mid_F) = (g^F_{i,j}\mid_F), \text{ and } \|(g^F_{i,j})\| \leq C\}.$$

Proof. The collection of finite subsets of X determines a directed set, ordered by inclusion. If we choose for each finite set F, functions $(g^F_{i,j})$ satisfying the conditions of the right hand set, then these functions define a net that converges BPW to $(f_{i,j})$ and hence, the right hand side is larger than the left. Conversely, given a net $(f^\lambda_{i,j})$ that converges pointwise to $(f_{i,j})$ and satisfies $\|(f^\lambda_{i,j})\| \leq C$ and any finite set $F = \{x_1, ..., x_k\}$, choose functions in $\mathcal{A}$ such that $f_i(x_j) = \delta_{i,j}$. If we let $A^\lambda_l = (f_{i,j}(x_l)) - (f^\lambda_{i,j}(x_l))$, then $(g^\lambda_{i,j}) = (f^\lambda_{i,j}) + A^\lambda_1 f_1 + \cdots A^\lambda_k f_k \in M_n(\mathcal{A})$ and is equal to $(f_{i,j})$ on F. Moreover, $\|(g^\lambda_{i,j})\| \leq \|(f^\lambda_{i,j})\| + \|A^\lambda_1\|_{M_n}\|f_1\|_{\mathcal{A}} + \cdots + \|A^\lambda_k\|_{M_n}\|f_k\|_{\mathcal{A}}$. Thus, given $\epsilon > 0$, since the functions $f_1, ..., f_k$ depend only on F, we may choose λ so that $\|(g^\lambda_{i,j})\| < C + \epsilon$. This shows the other inequality. □

Lemma 2.2.5. *Let $\mathcal{A}$ be an operator algebra of functions on the set X, then $\tilde{\mathcal{A}}$ equipped with the collection of norms on $M_n(\tilde{\mathcal{A}})$ given in Definition 2.2.3 is an operator algebra.*

Proof. It is clear from the definition of $\tilde{\mathcal{A}}$ that it is an algebra. Thus, it is enough to check that the axioms of BRS are satisfied by the algebra $\tilde{\mathcal{A}}$ equipped with the matrix norms given in the Definition 2.2.3.

If L and M are scalar matrices of appropriate sizes and $G \in M_n(\tilde{\mathcal{A}})$, then for $\epsilon > 0$ there exists $G_\lambda \in M_n(\mathcal{A})$ such that $\lim_\lambda G_\lambda(x) = G(x)$ for all $x \in X$ and $\sup_\lambda \|G_\lambda\|_{M_n(\mathcal{A})} \leq \|G\|_{M_n(\tilde{\mathcal{A}})} + \epsilon$. Since $\mathcal{A}$ is an operator space, $LG_\lambda M \in M_n(\mathcal{A})$ and $\|LG_\lambda M\|_{M_n(\mathcal{A})} \leq \|L\|\|G_\lambda\|_{M_n(\mathcal{A})}\|M\|$. Note that it follows that $\|LGM\|_{M_n(\tilde{\mathcal{A}})} \leq \|L\|\|G\|_{M_n(\tilde{\mathcal{A}})}\|M\|$, since $LG_\lambda M \to LGM$ pointwise and $\sup_\lambda \|LG_\lambda M\|_{M_n(\mathcal{A})} \leq \|L\|(\|G\|_{M_n(\tilde{\mathcal{A}})} + \epsilon)\|M\|$ for any $\epsilon > 0$.

If $G,\ H \in M_n(\tilde{\mathcal{A}})$, then for every $\epsilon > 0$ there exists $G_\lambda,\ H_\lambda \in M_n(\mathcal{A})$ such that $\lim_\lambda G_\lambda(x) = G(x)$ and $\lim_\lambda H_\lambda(x) = H(x)$ for every $x \in X$. Also, we have that $\sup_\lambda \|G_\lambda\|_{M_n(\mathcal{A})} \leq \|G\|_{M_n(\tilde{\mathcal{A}})} + \epsilon$ and $\sup_\lambda \|H_\lambda\|_{M_n(\mathcal{A})} \leq \|H\|_{M_n(\tilde{\mathcal{A}})} + \epsilon$.

Let $L = GH$ and $L_\lambda = G_\lambda H_\lambda$. Since $\mathcal{A}$ is matrix normed algebra, $L_\lambda \in M_n(\mathcal{A})$ and $\|L_\lambda\|_{M_n(\mathcal{A})} \leq \|G_\lambda\|_{M_n(\mathcal{A})}\|H_\lambda\|_{M_n(\mathcal{A})}$ for every λ. This implies that $\lim_\lambda L_\lambda(x) = L(x)$ and that

$$\|L\|_{M_n(\tilde{\mathcal{A}})} \leq \sup_\lambda \|L_\lambda\|_{M_n(\mathcal{A})} \leq \sup_\lambda \|G_\lambda\|_{M_n(\mathcal{A})} \sup_\lambda \|H_\lambda\|_{M_n(\mathcal{A})}.$$

This yields $\|L\|_{M_n(\tilde{\mathcal{A}})} \leq \|G\|_{M_n(\tilde{\mathcal{A}})}\|H\|_{M_n(\tilde{\mathcal{A}})}$, and so the multiplication is completely contractive.

Finally, to see that the L^∞ conditions are met, let $G \in M_n(\tilde{\mathcal{A}})$ and $H \in M_m(\tilde{\mathcal{A}})$. Given $\epsilon > 0$ there exist $G_\lambda \in M_n(\mathcal{A})$ and $H_\lambda \in M_m(\mathcal{A})$ such that $\lim_\lambda G_\lambda(x) = G(x)$, $\lim_\lambda H_\lambda(x) = H(x)$ and $\sup_\lambda \|G_\lambda\|_{M_n(\mathcal{A})} \leq \|G\|_{M_n(\tilde{\mathcal{A}})} + \epsilon,\ \sup_\lambda \|H_\lambda\|_{M_n(\mathcal{A})} \leq \|H\|_{M_n(\tilde{\mathcal{A}})} + \epsilon$.

Note that $G_\lambda \oplus H_\lambda \in M_{n+m}(\mathcal{A})$ and $\|G_\lambda \oplus H_\lambda\| = \max\{\|G_\lambda\|_{M_n(\mathcal{A})}, \|H_\lambda\|_{M_n(\mathcal{A})}\}$ for every λ which implies that $G \oplus H \in M_{n+m}(\tilde{\mathcal{A}})$, and

$$\begin{aligned}\|G \oplus H\|_{M_{n+m}(\tilde{\mathcal{A}})} \leq \sup_\lambda \|G_\lambda \oplus H_\lambda\| &= \sup_\lambda [\max\{\|G_\lambda\|_{M_n(\mathcal{A})}, \|H_\lambda\|_{M_m(\mathcal{A})}\}] \\ &= \max\{\sup_\lambda \|G_\lambda\|_{M_n(\mathcal{A})}, \sup_\lambda \|H^\lambda\|_{M_m(\mathcal{A})}\} \\ &\leq \max\{\|G\|_{M_n(\tilde{\mathcal{A}})} + \epsilon, \|H\|_{M_m(\tilde{\mathcal{A}})} + \epsilon\}.\end{aligned}$$

This shows that $\|G \oplus H\|_{M_{n+m}(\tilde{\mathcal{A}})} \leq \max\{\|G\|_{M_n(\tilde{\mathcal{A}})}, \|H\|_{M_m(\tilde{\mathcal{A}})}\}$, and so the L^∞ condition follows. This completes the proof of the result. □

Lemma 2.2.6. *If $\mathcal{A}$ is an operator algebra of functions on the set X, then $\tilde{\mathcal{A}}$ equipped with the norms of Definition 2.2.3 is a local operator algebra of functions on X. Moreover, for every $(f_{i,j}) \in M_n(\mathcal{A})$, $\|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} = \|(f_{i,j})\|_{M_n(\mathcal{A}_L)}$.*

Proof. It is clear from the definition of the norms on $\tilde{\mathcal{A}}$ that the identity map from $\mathcal{A}$ to $\tilde{\mathcal{A}}$ is completely contractive and thus $\mathcal{A} \subseteq \tilde{\mathcal{A}}$ as sets. This shows that $\tilde{\mathcal{A}}$ separates points of X and contains the constant functions.

Let $(f_{i,j}) \in M_n(\tilde{\mathcal{A}})$ and $\epsilon > 0$, then there exists a net $(f_{i,j}^\lambda) \in M_n(\mathcal{A})$ such that $\lim_\lambda (f_{i,j}^\lambda(x)) = (f_{i,j}(x))$ for each $x \in X$ and $\sup_\lambda \|(f_{i,j}^\lambda)\|_{M_n(\mathcal{A})} \leq \|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} + \epsilon$. Since $\mathcal{A}$ is an operator algebra of functions on the set X, we have that $\|(f_{i,j}^\lambda)\|_\infty \leq \|(f_{i,j}^\lambda)\|_{M_n(\mathcal{A})}$. Thus, $\sup_\lambda \|(f_{i,j}^\lambda)\|_\infty \leq \|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} + \epsilon$. Fix $z \in X$, then

$$\|(f_{i,j}(z))\| = \lim_\lambda \|(f_{i,j}^\lambda(z))\| \leq \sup_\lambda \|(f_{i,j}^\lambda)\|_\infty \leq \|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} + \epsilon.$$

By letting $\epsilon \to 0$ and taking the supremum over $z \in X$, we get that $\|(f_{i,j})\|_\infty \leq \|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})}$. Hence, $\tilde{\mathcal{A}}$ is an operator algebra of functions on the set X.

Set $\tilde{I}_F = \{f \in \tilde{\mathcal{A}} : f|_F \equiv 0\}$ and let $(f_{i,j}) \in M_n(\tilde{\mathcal{A}})$. Then, clearly $\sup_F \|(f_{i,j} + \tilde{I}_F)\|_{M_n(\tilde{\mathcal{A}}/\tilde{I}_F)} \leq \|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})}$. To see the other inequality, assume that $\sup_F \|(f_{i,j} + \tilde{I}_F)\| <$

1. Then for every finite $F \subseteq X$ there exists $(h^F_{i,j}) \in M_n(\tilde{\mathcal{A}})$ such that $(h^F_{i,j})|_F = (f^F_{i,j})|_F$ and $\sup_F \|h^F_{i,j}\| \leq 1$. Fix a set $F \subseteq X$ and $(h^F_{i,j}) \in M_n(\tilde{\mathcal{A}})$. Then for all finite $F' \subseteq X$ there exists $(k^{F'}_{i,j}) \in M_n(\mathcal{A})$ such that $(k^{F'}_{i,j})|_{F'} = (h^F_{i,j})|_{F'}$ and $\sup_{F'} \|k^{F'}_{i,j}\| \leq 1$.

In particular, let $F' = F$ then $(k^F_{i,j})|_F = (h^F_{i,j})|_F = (f_{i,j})|_F$ and $\sup_F \|k^F_{i,j}\| \leq 1$. Hence, $\|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} \leq 1$, and $\|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} \leq \sup_F \|(f_{i,j} + \tilde{I}_F)\|_{M_n(\tilde{\mathcal{A}}/\tilde{I}_F)}$. Thus, for every $(f_{i,j}) \in M_n(\mathcal{A})$,

$$\|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} = \sup_F \|(f_{i,j} + \tilde{I}_F)\|_{M_n(\tilde{\mathcal{A}}/\tilde{I}_F)}.$$

Note that for any $F \subseteq X$ we have $\|(f_{i,j} + \tilde{I}_F)\|_{M_n(\tilde{\mathcal{A}}/\tilde{I}_F)} \leq \|(f_{i,j} + I_F)\|_{M_n(\mathcal{A}/I_F)}$, since $I_F \subseteq \tilde{I}_F$. We claim that $\|(f_{i,j} + I_F)\| = \|(f_{i,j} + \tilde{I}_F)\|$ for every $(f_{i,j}) \in M_n(\mathcal{A})$, and for every finite subset $F \subseteq X$. To see the other inequality, let $(g_{i,j}) \in M_n(\tilde{I}_F)$. Then for $\epsilon > 0$ and $G \subseteq X$, we may choose $(hf^G_{i,j}) \in M_n(\mathcal{A})$ such that $(h^G_{i,j})|_G = (f_{i,j} + g_{i,j})|_G$ and $\sup_G \|(h^G_{i,j})\| \leq \|(f_{i,j} + g_{i,j})\| + \epsilon$. Hence, $\|(f_{i,j} + I_F)\| = \|(h^F_{i,j} + I_F)\| \leq \|(h^F_{i,j})\| \leq \|(f_{i,j} + g_{i,j})\| + \epsilon$. Since $\epsilon > 0$ was arbitrary, the equality follows.

Now it is clear that,

$$\|(f_{i,j})\|_{M_n(\tilde{\mathcal{A}})} = \sup_F \|(f_{i,j} + I_F)\| = \|(f_{i,j})\|_{M_n(\mathcal{A}_L)},$$

and so the result follows. □

Corollary 2.2.7. *If $\mathcal{A}$ is a BPW complete operator algebra then $\mathcal{A}_L = \tilde{\mathcal{A}}$ completely isometrically.*

Proof. Since $\mathcal{A}$ is BPW complete, $\mathcal{A} = \tilde{\mathcal{A}}$ as sets. But by Lemma 2.2.4, the norm defined on $\mathcal{A}_L$ agrees with the norm defined on $\tilde{\mathcal{A}}$. □

Remark 2.2.8. *In the view of the above corollary, we denote the norm on $\tilde{\mathcal{A}}$ by $\|.\|_L$. Note that $\mathcal{A} \subseteq_{cc} \mathcal{A}_L \subseteq_{ci} \tilde{\mathcal{A}}$ for every operator algebra of functions $\mathcal{A}$.*

Lemma 2.2.9. *If $\mathcal{A}$ is an operator algebra of functions on X, then $Ball(\mathcal{A}_L)$ is BPW dense in $Ball(\tilde{\mathcal{A}})$ and $\tilde{\mathcal{A}}$ is BPW complete, i.e., $\tilde{\tilde{\mathcal{A}}} = \tilde{\mathcal{A}}$.*

Proof. It can be easily checked that the statement is equivalent to showing that $\mathcal{A}_L$ is BPW dense in $\tilde{\mathcal{A}}$. We'll only prove that $\overline{\mathcal{A}_L}^{BPW} \subseteq \tilde{\mathcal{A}}$, since the other containment follows immediately by the definition of $\tilde{\mathcal{A}}$.

Let $\{f_\lambda\}$ be a net in $\mathcal{A}_L$ such that $f_\lambda \to f$ pointwise and $\sup_\lambda \|f_\lambda\|_{\mathcal{A}_L} < C$. Then for fixed $F \subseteq X$ and $\epsilon > 0$, there exists λ_F such that $|f_{\lambda_F}(z) - f(z)| < \epsilon$ for $z \in F$. Also since $\sup_\lambda \|f_\lambda\| < C$, there exists $g_{\lambda_F} \in I_F$ such that $\|f_{\lambda_F} + g_{\lambda_F}\| < C$. Note that the function $h_F = f_{\lambda_F} + g_{\lambda_F} \in \mathcal{A}$ satisfies $\|h_F\|_{\mathcal{A}} < C$, and $h_F \to f$ pointwise. Thus, $f \in \tilde{\mathcal{A}}$ and hence, $\mathcal{A}_L$ is BPW dense in $\tilde{\mathcal{A}}$. Finally, a similar argument yields that $\tilde{\mathcal{A}}$ is BPW complete. □

All the above lemmas can be summarized as the following theorem.

Theorem 2.2.10. *If $\mathcal{A}$ is an operator algebra of functions on X, then $\tilde{\mathcal{A}}$ is a BPW complete local operator algebra of functions on X which contains $\mathcal{A}_L$ completely isometrically as a BPW dense subalgebra.*

Definition 2.2.11. *Given an operator algebra of functions $\mathcal{A}$ on X, we call $\tilde{\mathcal{A}}$ the* **BPW completion of** $\mathcal{A}$.

2.2.3 Examples

In this section, we present a few examples to illustrate the concepts introduced in the earlier sections.

Example 2.2.12. *If $\mathcal{A}$ is a* uniform algebra, *then there exists a compact, Hausdorff space X, such that $\mathcal{A}$ can be represented as a subalgebra of $C(X)$ that separates points. If* *we endow $\mathcal{A}$ with the matrix-normed structure that it inherits as a subalgebra of $C(X)$, namely, $\|(f_{i,j})\| = \|(f_{i,j})\|_\infty \equiv \sup\{\|(f_{i,j})\|_{M_n} : x \in X\}$, then $\mathcal{A}$ is a local operator algebra of functions on X. Indeed, to achieve the norm, it is sufficient to take the supremum over all finite subsets consisting of one point. In this case the BPW completion $\tilde{\mathcal{A}}$ is completely isometrically isomorphic to the subalgebra of $\ell^\infty(X)$ consisting of functions that are bounded, pointwise limits of functions in $\mathcal{A}$.*

Example 2.2.13. *Let $\mathcal{A} = A(\mathbb{D}) \subseteq C(\mathbb{D}^-)$ be the subalgebra of the algebra of continuous functions on the closed disk consisting of the functions that are analytic on the open disk $\mathbb{D}$. Identifying $M_n(A(\mathbb{D})) \subseteq M_n(C(\mathbb{D}^-))$ as a subalgebra of the algebra of continuous functions from the closed disk to the matrices, equipped with the supremum norm, gives $A(\mathbb{D})$ the usual operator algebra structure. With this structure it can be regarded as a local operator algebra of functions on $\mathbb{D}$ or on $\mathbb{D}^-$. If we regard it as a local operator algebra of functions on $\mathbb{D}^-$, then $A(\mathbb{D}) \subsetneq \widetilde{A(\mathbb{D})}$. To see that the containment is strict, note that $f(z) = (1+z)/2 \in A(\mathbb{D})$ and $f^n(z) \to \chi_{\{1\}}$, the characteristic function of the singleton $\{1\}$.*

However, if we regard $A(\mathbb{D})$ as a local operator algebra of functions on $\mathbb{D}$, then its BPW completion $\widetilde{A(\mathbb{D})} = H^\infty(\mathbb{D})$, the bounded analytic functions on the disk, with its usual operator structure.

Example 2.2.14. *Let $X = \epsilon\mathbb{D}$, $0 < \epsilon < 1$ and $\mathcal{A} = \{f|_X : f \in H^\infty(\mathbb{D})\}$. If we endow $\mathcal{A}$ with the matrix-normed structure on $H^\infty(\mathbb{D})$, then $\mathcal{A}$ is an operator algebra of functions on X. Also, it can be verified that $\mathcal{A}$ is a local operator algebra of functions and that $\mathcal{A} = \tilde{\mathcal{A}}$. Indeed, if $F = (f_{i,j}) \in M_n(\mathcal{A})$ with $\|(f_{i,j} + I_Y)\|_\infty < 1$ for all finite subset $Y \subseteq X$, then there exists $H_Y \in M_n(\mathcal{A})$ such that $\|H_Y\|_\infty \leq 1$ and $H_Y \to F$ pointwise on X. Note by Montel's theorem [30] there exist a subnet $H_{Y'}$ and $G \in M_n(H^\infty(\mathbb{D}))$ such that $\|G\|_\infty \leq 1$*

and $H_{Y'} \to G$ uniformly on compact subsets of $\mathbb{D}$. Thus, by the identity theorem $F \equiv G$ on $\mathbb{D}$. Hence, $\|F\|_{M_n(\mathcal{A})} \leq 1$ and so $\mathcal{A}$ is a local operator algebra. A similar argument shows that if f is a BPW limit on X, then there exists $g \in H^\infty(\mathbb{D})$ such that $g|_X = f$, and so $\mathcal{A}$ is BPW complete. By Lemma 2.2.9, $\tilde{\mathcal{A}} = \mathcal{A}$ completely isometrically.

Example 2.2.15. *Let $\mathcal{A} = H^\infty(\mathbb{D})$ but endowed with a new norm. Since for every fixed $b > 1$, the spectrum of $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is trivially contained in $\mathbb{D}$, thus $F(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix})$ can be defined for every $F \in M_{\mathcal{A}}$ by using functional calculus. For every $F \in M_n(\mathcal{A})$, set $\|F\| = \max\{\|F\|_\infty, \|F(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix})\|\}$. It can be easily verified that $\mathcal{A}$ is a BPW complete operator algebra of functions. However, we also claim that $\mathcal{A}$ is local. To prove this we proceed by contradiction. Suppose there exists $F = (f_{i,j}) \in M_n(H^\infty(\mathbb{D}))$ such that $\|F\| > 1 > c$, where $c = \sup_Y \|(f_{i,j} + I_Y)\|$.*

In this case, $\|F\| = \|F(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix})\|$, since $\|(f_{i,j} + I_Y)\| = \|F(\lambda)\|$ when $Y = \{\lambda\}$. Let $\epsilon = \frac{1-c}{4b}$ and $Y = \{0, \epsilon\} \subseteq \mathbb{D}$, then there exists $G \in M_n(H^\infty(\mathbb{D}))$ such that $G|_Y = 0$ and $\|F + G\| < \frac{1+c}{2}$.

Thus, we can write $B_Y(z) = \frac{z-\epsilon}{1-\bar{\epsilon}z}$, so that $G(z) = zB_Y(z)H(z)$, for some $H \in M_n(H^\infty)$. It follows that $\|H\|_\infty < 2$, since $\|G\|_\infty < 2$. We now consider

$$1 < \left\| F(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}) \right\| \leq \left\| (F+G)(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}) \right\| + \left\| G(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}) \right\|$$

$$\leq \frac{1+c}{2} + \left\| \begin{pmatrix} 0 & bG'(0) \\ 0 & 0 \end{pmatrix} \right\| = \frac{1+c}{2} + b|B_Y(0)|\|H(0)\|$$

$$\leq \frac{1+c}{2} + 2b\epsilon = \frac{1+c}{2} + 2b\frac{1-c}{4b} = 1,$$

which is a contradiction.

Example 2.2.16. *This is an example of a non-local algebra that arises from boundary behavior. Let* $\mathcal{A} = A(\mathbb{D})$ *equipped with the family of matrix norms*

$$\|F\| = \max\{\|F\|_\infty, \|F(\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix})\|\}, \ F \in M_n(\mathcal{A})$$

where $F(\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix})$ *is defined by using power series exapansion. Note that when* $F(1) = F(-1)$, *then* $\|F\| = \|F\|_\infty$. *It is easy to check that* $\mathcal{A}$ *is an operator algebra of functions on the set* $\mathbb{D}$ *that is not BPW complete. Also, it can be verified that* $\mathcal{A}$ *is not local. To see this, note that* $\|z\| = \left\| \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\| > 1$. *Fix* $\alpha > 0$, *such that* $1 + 2\alpha < \|z\|$. *For each* $Y = \{z_1, z_2, \ldots, z_n\}$, *we define* $B_Y(z) = \Pi_{i=1}^n (\frac{z - z_i}{1 - \bar{z}_i z})$ *and choose* $h \in \mathcal{A}$ *such that* $h(1) = -\overline{B_Y(1)}, \ h(-1) = \overline{B_Y(-1)}$, *and* $\|h\|_\infty \leq 2$.

Let $g(z) = z + B_Y(z)h(z)\alpha$, *then* $g \in \mathcal{A}, \ g(1) = g(-1)$ *and* $g|_Y = z|_Y$. *Hence,* $\|\pi_Y(z)\| = \|\pi_Y(g)\| \leq \|g\| = \|g\|_\infty \leq 1 + 2\alpha < \|z\|$. *Thus, since* α *was arbitrary,* $\sup_{Y \subseteq \mathbb{D}} \|\pi_Y(z)\| = 1 < \|z\|$ *and hence* $\mathcal{A}$ *is not local.*

Example 2.2.17. *This example shows that one can easily build non-local algebras by adding "values" outside of the set* X. *Let* $\mathcal{A}$ *be the algebra of polynomials regarded as functions on the set* $X = \mathbb{D}$. *Then* $\mathcal{A}$ *endowed with the matrix-normed structure as* $\|(p_{i,j})\| = \max\{\|(p_{i,j})\|_\infty, \|(p_{i,j}(2))\|\}$, *is an operator algebra of functions on the set* X. *To see that* $\mathcal{A}$ *is not local, let* $p \in \mathcal{A}$ *be such that* $\|p\|_\infty < |p(2)|$. *For each finite subset* $Y = \{z_1, \ldots, z_n\}$ *of* X, *let* $h_Y(z) = \Pi_{i=1}^n (z - z_i)$ *and* $g_Y(z) = p(z) - \alpha h_Y(z) p(2)$, *where* $\alpha = \frac{|p(2)| - \|p\|_\infty}{2|p(2)| \|h_Y\|_\infty} > 0$. *Note that* $\|g_Y\| \leq (1 - \alpha)|p(2)|$ *and* $g_Y|_Y = p|_Y$. *Hence,* $\|\pi_Y(p)\| = \|\pi_Y(g_Y)\| \leq \|g_Y\| \leq (1 - \alpha)|p(2)| < \|p\|$. *It follows that* $\mathcal{A}$ *is not local.*

Finally, observe that in this case, $\mathcal{A}$ *cannot be BPW complete. For example, if we take* $p_n = \frac{1}{3} \sum_{i=0}^n (\frac{z}{3})^i \in \mathcal{A}$ *then* $p_n(z) \to f(z) = \frac{1}{3 - z}$ *for* $z \in \mathbb{D}$ *and* $\|p_n\| < \|f\|$, *which implies* *that* $\mathcal{A}_L \subsetneq \tilde{A}$.

Example 2.2.18. *It is still an open problem as to whether or not every unital contractive, homomorphism* $\rho : H^\infty(\mathbb{D}) \to B(\mathcal{H})$ *is completely contractive. For a recent discussion of this problem, see [64]. Let's assume that* ρ *is a contractive homomorphism that is not completely contractive. Let* $\mathcal{B} = H^\infty(\mathbb{D})$*, but endow it with the family of matrix-norms given by,*

$$|||(f_{i,j})||| = max\{\|(f_{i,j})\|_\infty, \|(\rho(f_{i,j}))\|\}.$$

Note that $|||f||| = \|f\|_\infty$*, for* $f \in \mathcal{B}$.

It is easily checked that $\mathcal{B}$ *is a BPW complete operator algebra of functions on* $\mathbb{D}$*. However, since every contractive homomorphism of* $A(\mathbb{D})$ *is completely contractive, we have that for* $(f_{i,j}) \in M_n(A(\mathbb{D})), |||(f_{i,j})||| = \|(f_{i,j})\|_\infty$*. If* $Y = \{x_1, ..., x_k\}$ *is a finite subset of* $\mathbb{D}$ *and* $F = (f_{i,j}) \in M_n(\mathcal{B})$*, then there is* $G = (g_{i,j}) \in M_n(A(\mathbb{D}))$*, such that* $F(x) = G(x)$ *for all* $x \in Y$*, and* $\|G\|_\infty = \|F\|_\infty$*. Hence,* $\|\pi_Y^{(n)}(F)\| \leq \|F\|_\infty$*. Thus,* $\sup_Y \|\pi_Y^{(n)}(F)\| = \|F\|_\infty$*. It follows that* $\mathcal{B}$ *is not local and that* $\tilde{\mathcal{B}} = \mathcal{B}_L = H^\infty(\mathbb{D})$*, with its usual supremum norm operator algebra structure.*

In particular, if there does exist a contractive but not completely contractive representation of $H^\infty(\mathbb{D})$*, then we have constructed an example of a non-local BPW complete operator algebra of functions on* $\mathbb{D}$.

Note that any operator algebra of functions $\mathcal{A}$ satisfies $\mathcal{A} \subseteq \mathcal{A}_L \subseteq \tilde{\mathcal{A}}$. The above set of examples provides us a good analysis of the above equation and covers all possible combination of example. We close this section by summarizing it in the following table.

Type	Relation	Example
Local and BPW complete	$\mathcal{A} =_{ci} \mathcal{A}_L =_{ci} \tilde{\mathcal{A}}$	Examples 2.2.14, 2.2.15
Local but not BPW complete	$\mathcal{A} =_{ci} \mathcal{A}_L \subsetneq_{ci} \tilde{\mathcal{A}}$	Examples 2.2.12, 2.2.13
Non-local but BPW complete	$\mathcal{A} \subsetneq_{cc} \mathcal{A}_L =_{ci} \tilde{\mathcal{A}}$	Examples 2.2.18
Non-local and not BPW complete	$\mathcal{A} \subsetneq_{cc} \mathcal{A}_L \subsetneq_{ci} \tilde{\mathcal{A}}$	Examples 2.2.16, 2.2.17

2.3 A Characterization of Local OPAF

The main goal of this section is to prove that every BPW complete local operator algebra of functions is completely isometrically isomorphic to the algebra of multipliers on a reproducing kernel Hilbert space of vector-valued functions. Moreover, we will show that every such algebra is a dual operator algebra in the precise sense of [23]. We will then prove that for such BPW algebras, weak*-convergence and BPW convergence coincide on bounded balls. But before proving them, we need a short guide of terminology of some concepts from functional analysis.

First we mention a few basic facts and some terminology from the theory of vector-valued reproducing kernel Hilbert spaces. Given a set X and a Hilbert space $\mathcal{H}$, then by a *reproducing kernel Hilbert space of* $\mathcal{H}$*-valued functions on* X, we mean a vector space $\mathcal{L}$ of $\mathcal{H}$-valued functions on X that is equipped with a norm and an inner product that makes it a Hilbert space and which has the property that for every $x \in X$, the evaluation map $E_x : \mathcal{L} \to \mathcal{H}$, is a bounded, linear map. Recall that given a Hilbert space $\mathcal{H}$, a matrix of operators, $T = (T_{i,j}) \in M_k(B(\mathcal{H}))$ is regarded as an operator on the Hilbert space $\mathcal{H}^{(k)} \equiv \mathcal{H} \otimes \mathbb{C}^k$, which is the direct sum of k copies of $\mathcal{H}$. A function $K : X \times X \to B(\mathcal{H})$, where H is a Hilbert space, is called a *positive definite operator-valued function on* X, provided that for every finite set of (distinct)points $\{x_1, ..., x_k\}$ in X, the operator-valued matrix, $(K(x_i, x_j))$ is positive semidefinite. Given a reproducing kernel Hilbert space $\mathcal{L}$ of $\mathcal{H}$-valued functions, if we set $K(x,y) = E_x E_y^*$, where $E_x : \mathcal{L} \to \mathcal{H}$ is the point evaluation map, then K is positive definite and is called the *reproducing kernel of* $\mathcal{L}$. The proof of this involves direct matrix calculation. On the contrary, the proof of the converse to this fact is not that straightforward and is generally called *Moore's theorem,* which states that given any positive definite operator-valued function $K : X \times X \to B(\mathcal{H})$, then there exists a unique reproducing kernel Hilbert space of $\mathcal{H}$-valued functions on X, such that $K(x,y) = E_x E_y^*$. We will denote this space by $\mathcal{L}(K, \mathcal{H})$.

Given $v,\ w \in \mathcal{H}$, we let $v \otimes w^* : \mathcal{H} \to \mathcal{H}$ denote the rank one operator given by $(v \otimes w^*)(h) = \langle h, w \rangle v$. A function $g : X \to \mathcal{H}$ belongs to $\mathcal{L}(K, \mathcal{H})$ if and only if there exists a constant $C > 0$ such that the function

$$C^2 K(x,y) - g(x) \otimes g(y)^*$$

is positive definite. In which case the norm of g is the least such constant. Finally, given any reproducing kernel Hilbert space $\mathcal{L}$ of $\mathcal{H}$-valued functions with reproducing kernel K, a function $f : X \to \mathbb{C}$ is called a (scalar) *multiplier* provided that for every $g \in \mathcal{L}$, the function $fg \in \mathcal{L}$. In this case it follows by an application of the closed graph theorem that the map $M_f : \mathcal{L} \to \mathcal{L}$, defined by $M_f(g) = fg$, is a bounded, linear map. The set of all multipliers is denoted by $\mathcal{M}(K)$ and is easily seen to be an algebra of functions on X and a subalgebra of $B(\mathcal{L})$. The reader can find proofs of the above facts in [26] and [9]. Also, we refer to the fundamental work of Pedrick [65] for further treatment of vector-valued reproducing kernel Hilbert spaces. Another good source is [5].

Given a normed space Y, we define the dual of Y as the set of all linear functionals on Y and denote it by Y^*. Then the weak*-topology on Y^* is the smallest topology on Y^* that makes all the linear functionals in $\{F_y : y \in Y\}$ continuous. Thus a net $\phi_\lambda \to \phi$

in Y^* in the weak*-topology if and only if $\phi_\lambda(y) \to \phi(y)$ for all $y \in Y$. A space is called weak*-closed if it is closed in the weak*-topology. We record an important and a very useful theorem concerning weak*-topology called *Krein-Smulian theorem.* There are many parts to this theorem but we only need the following.

Theorem 2.3.1. *Let Y be a dual Banach space, and let V be a linear subspace of Y. Then V is weak*-closed in Y if and only if Ball(V) is closed in the weak*-topology on Y. In this case, V is also a dual Banach space.*

The proof of the above can be found in many standard texts on functional analysis. We refer the reader to [23, Section 1.4]. We are now in a position to state and prove the following result about the multiplier algebras of a reproducing kernel Hilbert spaces.

Lemma 2.3.2. *Let $\mathcal{L}$ be a reproducing kernel Hilbert space of $\mathcal{H}$-valued functions with reproducing kernel $K : X \times X \to B(\mathcal{H})$. Then $\mathcal{M}(K) \subseteq B(\mathcal{L})$ is a weak*-closed subalgebra.*

Proof. It is enough to show that the unit ball is weak*-closed by the Krein-Smulian theorem. So let $\{M_{f_\lambda}\}$ be a net of multipliers in the unit ball of $B(\mathcal{L})$ that converges in the weak*-topology to an operator T. We must show that T is a multiplier.

Let $x \in X$ be fixed and assume that there exists $g \in \mathcal{L}$, with $g(x) = h \neq 0$. Then $\langle Tg, E_x^*h\rangle_{\mathcal{L}} = \lim_\lambda \langle M_{f_\lambda} g, E_x^* h\rangle_{\mathcal{L}} = \lim_\lambda \langle E_x(M_{f_\lambda} g), h\rangle_{\mathcal{H}} = \lim_\lambda f_\lambda(x)\|h\|^2$. This shows that at every such x the net $\{f_\lambda(x)\}$ converges to some value. Set $f(x)$ equal to this limit and for all other x's set $f(x) = 0$. We claim that f is a multiplier and that $T = M_f$.

Note that if $g(x) = 0$ for every $g \in \mathcal{L}$, then $E_x = E_x^* = 0$. Thus, we have that for any $g \in \mathcal{L}$ and any $h \in \mathcal{H}$, $\langle E_x(Tg), h\rangle_{\mathcal{H}} = \lim_\lambda \langle E_x(M_{f_\lambda} g), h\rangle_{\mathcal{H}} = \lim_\lambda f_\lambda(x)\langle g(x), h\rangle_{\mathcal{H}} = f(x)\langle g(x), h\rangle_{\mathcal{H}}$. Since this holds for every $h \in \mathcal{H}$, we have that $E_x(Tg) = f(x)g(x)$, and so $T = M_f$ and f is a multiplier. $\square$

In a fashion similar to operator algebras, by an operator space we mean a space that has both a vector space structure and matrix norm structure that satisfies Ruan's axioms. An operator space V is said to be a *dual operator space* if V is completely isometrically isomorphic to the operator space dual Y^* of an operator space Y. The reader can find the proof of the fact that "the dual operator spaces and weak*-closed subspaces of bounded operators on a Hilbert space are essentially the same thing" in [23, Section 1.4]. Thus, every weak*-closed subspace $V \subseteq B(\mathcal{H})$ has a predual and it is the operator space dual of this predual. Also, if an abstract operator algebra is the dual of an operator space, then it can be represented completely isometrically and weak*-continuously as a weak*-closed subalgebra of the bounded operators on some Hilbert space. For this reason an operator algebra that has a predual as an operator space is called a *dual operator algebra.* See [23] for the proofs of these facts. Thus, in summary, the above lemma shows that every multiplier algebra is a dual operator algebra in the sense of [23].

Theorem 2.3.3. *Let $\mathcal{L}$ be a reproducing kernel Hilbert space of $\mathcal{H}$-valued functions with reproducing kernel $K : X \times X \to B(\mathcal{H})$ and let $\mathcal{M}(K) \subseteq B(\mathcal{L})$ denote the multiplier algebra, endowed with the operator algebra structure that it inherits as a subalgebra. If $K(x,x) \neq 0$, for every $x \in X$ and $\mathcal{M}(K)$ separates points on X, then $\mathcal{M}(K)$ is a BPW complete local dual operator algebra of functions on X.*

Proof. The multiplier norm of a given matrix-valued function $F = (f_{i,j}) \in M_n(\mathcal{M}(K))$ is the least constant C such that

$$((C^2 I_n - F(x_i)F(x_j)^*) \otimes K(x_i, x_j)) \geq 0,$$

for all sets of finitely many points, $Y = \{x_1, ..., x_k\} \subseteq X$. Applying this fact to a set consisting of a single point, we have that $(C^2 I_n - F(x)F(x)^*) \otimes K(x,x) \geq 0$, and it follows that $C^2 I_n - F(x)F(x)^* \geq 0$. Thus,

$\|F(x)\| \leq C = \|F\|$ and we have that point evaluations are completely contractive on $\mathcal{M}(K)$. Since $\mathcal{M}(K)$ contains the constants and separates points by hypotheses, it is an operator algebra of functions on X.

Suppose that $\mathcal{M}(K)$ was not local, then there would exist $F \in M_n(\mathcal{M}(K))$, and a real number C, such that $\sup_Y \|\pi_Y^{(n)}\| < C < \|F\|$. Then for each finite set $Y = \{x_1, ..., x_k\}$ we could choose $G \in M_n(\mathcal{M}(K))$, with $\|G\| < C$, and $G(x) = F(x)$, for every $x \in Y$. But then we would have that $((C^2 I_n - F(x_i)F(x_j)^*) \otimes K(x_i, x_j)) = ((C^2 I_n - G(x_i)G(x_j)^*) \otimes K(x_i, x_j)) \geq 0$, and since Y was arbitrary, $\|F\| \leq C$, a contradiction. Thus, $\mathcal{M}(K)$ is local.

Finally, assume that $f_\lambda \in \mathcal{M}(K)$, is a net in $\mathcal{M}(K)$, with $\|f_\lambda\| \leq C$, and $\lim_\lambda f_\lambda(x) = f(x)$, pointwise. If $g \in \mathcal{L}$ with $\|g\|_{\mathcal{L}} = M$, then

$$(MC)^2 K(x, y) - f_\lambda(x)g(x) \otimes (f_\lambda(y)g(y))^*$$

is positive definite. By taking pointwise limits, we obtain that $(MC)^2 K(x, y) - f(x)g(x) \otimes (f(y)g(y))^*$ is positive definite. From the earlier characterization of functions in $\mathcal{L}$ and their norms in a reproducing kernel Hilbert space, this implies that $fg \in \mathcal{L}$, with $\|fg\|_{\mathcal{L}} \leq MC$. Hence, $f \in \mathcal{M}(K)$ with $\|M_f\| \leq C$. Thus, $\mathcal{M}(K)$ is BPW complete. □

In general, $\mathcal{M}(K)$ need not separate points on X. In fact, it is possible that $\mathcal{L}$ does not separate points and if $g(x_1) = g(x_2)$, for every $g \in \mathcal{L}$, then necessarily $f(x_1) = f(x_2)$ for every $f \in \mathcal{M}(K)$.

Following [61], we call $\mathcal{C}$ a *concrete k-idempotent operator algebra,* provided that there are k operators, $\{E_1, ..., E_k\}$ on some Hilbert space $\mathcal{H}$, such that $E_i E_j = E_j E_i = \delta_{i,j} E_i$, $I = E_1 + \cdots + E_k$ and $\mathcal{C} = \text{span}\{E_1, ..., E_k\}$. Recall, if C is an abstract operator algebra then can be represented on some Hilbert space $\mathcal{K}$ via a completely isometric homomorphism $\pi : \mathcal{C} \to B(\mathcal{K})$. We call C an *abstract k-idempotent operator algebra* if the image of C under the map π is a concrete k-idempotent operator algebra. We shall drop the term *concrete* and *abstract* whenever it is clear from the context.

Proposition 2.3.4. *Let $\mathcal{C} = span\{E_1, ..., E_k\}$ be a k-idempotent operator algebra on the Hilbert space $\mathcal{H}$, let $Y = \{x_1, ..., x_k\}$ be a set of k distinct points and define $K : Y \times Y \to B(\mathcal{H})$ by $K(x_i, x_j) = E_i E_j^*$. Then K is positive definite and $\mathcal{C}$ is completely isometrically isomorphic to $\mathcal{M}(K)$ via the map that sends $a_1E_1 + \cdots + a_kE_k$ to the multiplier $f(x_i) = a_i$.*

Proof. It is easily checked that K is positive definite. We first prove that the map is an isometry. Given $B = \sum_{i=1}^{k} a_i \otimes E_i \in \mathcal{C}$, let $f : Y \to \mathbb{C}$ be defined by $f(x_i) = a_i$. We have that $f \in \mathcal{M}(K)$ with $\|f\| \leq C$ if and only if $P = ((C^2 - f(x_i)f(x_j)^*)K(x_i, x_j))$ is positive semidefinite in $B(\mathcal{H}^{(k)})$. Let $v = e_1 \otimes v_1 + \cdots e_k \otimes v_k \in \mathcal{H}^{(k)}$, let $h = \sum_{j=1}^{k} E_j^* v_j$ and note that $E_j^* h = E_j^* v_j$. Finally, set $h = \sum_{i=1}^{k} h_i$. Thus,

$$\begin{aligned} \langle Pv, v\rangle = \sum_{i,j=1}^{k} (C^2 - a_i\bar{a}_j)\langle E_iE_j^* v_j, v_i\rangle &= \sum_{i,j=1}^{k} (C^2 - a_i\bar{a}_j)\langle E_j^* h, E_i^* h\rangle \\ &= C^2\|h\|^2 - \langle B^*h, B^*h\rangle = C^2\|h\|^2 - \|B^*h\|^2. \end{aligned}$$

Hence, $\|B\| \leq C$ implies that P is positive and so $\|M_f\| \leq \|B\|$.

For the converse, given any h let $v = \sum_{j=1}^{k} e_j \otimes E_j^* h$, and note that $\langle Pv, v\rangle \geq 0$, implies that $\|B^*h\| \leq C$, and so $\|B\| \leq \|M_f\|$. The proof of the complete isometry is similar but notationally cumbersome. □

Theorem 2.3.5. *Let $\mathcal{A}$ be an operator algebra of functions on the set X then there exist a Hilbert space, $\mathcal{H}$ and a positive definite function $K : X \times X \to B(\mathcal{H})$ such that $\mathcal{M}(K) = \tilde{\mathcal{A}}$ completely isometrically.*

Proof. Let Y be a finite subset of X. Since $\mathcal{A}/I_Y$ is a $|Y|$-idempotent operator algebra, by the above lemma, there exists a vector-valued kernel K_Y such that $\mathcal{A}/I_Y = \mathcal{M}(K_Y)$

completely isometrically.

Define

$$\widetilde{K_Y}(x,y) = \begin{cases} K_Y(x,y) & \text{when } (x,y) \in Y \times Y, \\ 0 & \text{when } (x,y) \notin Y \times Y \end{cases}$$

and set $K = \sum_Y \oplus \widetilde{K_Y}$, where the direct sum is over all finite subsets of X.

It is easily checked that K is positive definite. Let $f \in M_n(\mathcal{M}(K))$ with $\|M_f\| \leq 1$, which is equivalent to $((I_n - f(x)f(y)^*) \otimes K(x,y))$ being positive definite. This is in turn equivalent to $((I_n - f(x)f(y)^*) \otimes K_Y(x,y))$ being positive definite for every finite subset Y of X. This last condition is equivalent to the existence for each such Y of some $f_Y \in M_n(\mathcal{A})$ such that $\|\pi_Y(f_Y)\| \leq 1$ and $f_Y = f$ on Y. The net of functions $\{f_Y\}$ then converges BPW to f. Hence, $f \in \tilde{\mathcal{A}}$ with $\|f\|_L \leq 1$. This proves that $M_n(\mathcal{M}(K)) = M_n(\tilde{\mathcal{A}})$ isometrically, for every n, and the result follows. □

The above result draws a connection with the concept of *realizable Banach algebras*, introduced by Agler-McCarthy [5, Chapter 13] to study Pick's problem for algebras that are not multiplier algebras of Pick kernels. They call a commutative Banach algebra $\mathcal{A}$ of functions on the set X, *realizable* if there is a set of kernels $\{k^\alpha : \alpha \in I\}$ on X such that $\mathcal{A} = \bigcap_{\alpha \in I} \mathrm{Mult}(\mathcal{H}_{k^\alpha})$ and $\|\phi\|_{\mathcal{A}} = \sup_{\alpha \in I} \|\phi\|_{\mathrm{Mult}(\mathcal{H}_{k^\alpha})}$. It is clear that the algebras that are local and BPW complete are examples of *realizable algebras.*

Corollary 2.3.6. *Every BPW complete local operator algebra of functions is a dual operator algebra.*

Proof. In this case we have that $\mathcal{A} = \tilde{\mathcal{A}} = \mathcal{M}(K)$ completely isometrically. By Lemma 2.3.2, this latter algebra is a dual operator algebra. □

The above result gives a weak*-topology to a local operator algebra of functions $\mathcal{A}$ by using the identification $\mathcal{A} \subseteq \tilde{\mathcal{A}} = \mathcal{M}(K)$ and taking the weak*-topology of $\mathcal{M}(K)$. The following proposition proves that convergence of bounded nets in this weak*-topology on $\mathcal{A}$ is same as BPW convergence.

Proposition 2.3.7. *Let $\mathcal{A}$ be a local operator algebra of functions on the set X. Then the net $(f_\lambda)_\lambda \in Ball(\mathcal{A})$ converges in the weak*-topology if and only if it converges pointwise on X.*

Proof. Let $\mathcal{L}$ denote the reproducing kernel Hilbert space of $\mathcal{H}$-valued functions on X with kernel K for which $\tilde{\mathcal{A}} = \mathcal{M}(K)$. Recall that if $E_x : \mathcal{L} \to \mathcal{H}$, is the linear map given by evaluation at x, then $K(x,y) = E_x E_y^*$. Also, if $v \in \mathcal{H}$, and $h \in \mathcal{L}$, then $\langle h, E_x^* v\rangle_{\mathcal{L}} = \langle h(x), v\rangle_{\mathcal{H}}$.

First assume that the net $(f_\lambda)_\lambda \in Ball(\mathcal{A})$ converges to f in the weak*-topology. Using the identification of $\tilde{\mathcal{A}} = \mathcal{M}(K)$, we have that the operators M_{f_λ} of multiplication by f_λ, converge in the weak*-topology of $B(\mathcal{L})$ to M_f. Then for any $x \in X, h \in \mathcal{L}, v \in \mathcal{H}$, we have that

$$f_\lambda(x)\langle h(x), v\rangle_{\mathcal{H}} = \langle f(x)h(x), v\rangle_{\mathcal{H}} = \langle M_{f_\lambda} h, E_x^* v\rangle_{\mathcal{L}} \longrightarrow \langle M_f h, E_x^* v\rangle_{\mathcal{L}} = f(x)\langle h(x), v\rangle_{\mathcal{H}}.$$

Thus, if there is a vector in $\mathcal{H}$ and a vector in $\mathcal{L}$ such that $\langle h(x), v\rangle_{\mathcal{H}} \neq 0$, then we have that $f_\lambda(x) \to f(x)$. It is readily seen that such vectors exist if and only if $E_x \neq 0$, or equivalently, $K(x,x) \neq 0$. But this follows from the construction of K as a direct sum of positive definite functions over all finite subsets of X. For fixed $x \in X$ and the one element subset $Y_0 = \{x\}$, we have that the 1-idempotent algebra $\mathcal{A}/I_{Y_0} \neq 0$ and so $K_{Y_0}(x,x) \neq 0$, which is one term in the direct sum for $K(x,x)$.

Conversely, assume that $\|f_\lambda\| < K$, for all λ and $f_\lambda \to f$ pointwise on X. We must prove that $M_{f_\lambda} \to M_f$ in the weak*-topology on $B(\mathcal{L})$. But since this is a bounded net of operators, it will be enough to show convergence in the weak operator topology and arbitrary vectors can be replaced by vectors from a spanning set. Thus, it will be enough to show that for $v_1, v_2 \in \mathcal{H}$ and $x_1, x_2 \in X$, we have that $\langle M_{f_\lambda} E_{x_1}^* v_1, E_{x_2}^* v_2 \rangle_{\mathcal{L}} \to \langle M_f E_{x_1}^* v_1, E_{x_2}^* v_2 \rangle_{\mathcal{L}}$. But we have,

$$\langle M_{f_\lambda} E_{x_1}^* v_1, E_{x_2}^* v_2 \rangle_{\mathcal{L}} = \langle E_{x_2}(M_{f_\lambda} E_{x_1}^* v_1), v_2 \rangle_{\mathcal{H}} = f_\lambda(x_2) \langle K(x_2, x_1) v_1, v_2 \rangle_{\mathcal{H}}$$
$$\longrightarrow f(x_2) \langle K(x_2, x_1) v_1, v_2 \rangle_{\mathcal{H}} = \langle M_f E_{x_1}^* v_1, E_{x_2}^* v_2 \rangle_{\mathcal{L}},$$

and the result follows. □

Corollary 2.3.8. *The ball of a local operator algebra of functions is weak*-dense in the ball of its BPW completion.*

2.4 Residually Finite Dimensional Operator Algebras

A C^*-algebra is a Banach algebra A (complete normed algebra for which the product is contractive) having an involution $*$ (that is, conjugate linear map into itself satisfying $x^{**} = x$ and $(xy)^* = y^* x^*$ that satisfies $\|x^* x\| = \|x\|^2$ for all $x \in A$. The celebrated theorem of Gelfand-Naimark-Segal states that every C^*-algebra is isometrically $*$-isomorphic to a subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. It is a standard fact in a C^*-algebra theory that $*$-homomophisms are contractive, in fact, they are completely contractive since amplification of a $*$-homomorphism is also a $*$-homomorphism. Thus, for every $(a_{i,j}) \in M_n(A)$, we have that

$$\|(a_{i,j})\|_{M_n(A)} = \sup\{\|(\pi(a_{i,j}))\|\}$$

where the supremum is taken over all $*$-homomorphisms $\pi : A \to B(\mathcal{H})$ and all Hilbert spaces $\mathcal{H}$.

The notion of residually finite dimensional C^*-algebra has been around since quite some time now and has been useful for numerous reasons; they have been widely used as a tool in studying approximation theory of quasidiagonal C^*-algebras and various other classes of C^*-algebras. Also, they have been quite useful in the theory of groups due to its immense application.

Definition 2.4.1. *A C^*-algebra A is called* residually finite-dimensional*, abbreviated as RFD, if it has a separating family of finite-dimensional representations, that is, a family of $*$-homomorphisms into matrix algebras.*

Thus, to achieve a norm of an element of a RFD C^*-algebra A, it is enough to take the supremum over only finite dimensional representations. For every $(a_{i,j}) \in M_n(A)$, we have that

$$\|(a_{i,j})\|_{M_n(A)} = \sup_{\pi}\{\|(\pi(a_{i,j}))\|\}$$

where the supremum is taken over all finite dimensional $*$-homomorphisms, $\pi : A \to M_k$ and all k. More details on these algebras can found in [42], [22], [33],and [13].

In this section we give a natural and obvious way of extending this notion to general operator algebras. In connection with the theory of operator algebras described in earlier sections, we will show that there exists a large class of examples of RFD operator algebras.

Recall from Section 1.1, the BRS Theorem 2.1.1 states that every operator algebra $\mathcal{A}$ can be represented completely isometrically on a Hilbert space. Thus, for every element $(a_{i,j}) \in M_n(\mathcal{A})$, we have that

$$\|(a_{i,j})\|_{M_n(\mathcal{A})} = \sup\{\|(\pi(a_{i,j}))\|\}$$

where the supremum is taken over all completely contractive homomorphisms $\pi : \mathcal{A} \to B(\mathcal{H})$ and all Hilbert spaces $\mathcal{H}$. This together with the consequence of GNS theorem for

RFD C^*-algebras motivates the following natural definition of *RFD operator algebras.*

Definition 2.4.2. *An operator algebra, $\mathcal{B}$ is called* **RFD** *if for every n and for every $(b_{i,j}) \in M_n(\mathcal{B})$, $\|(b_{i,j})\| = \sup\{\|(\pi(b_{i,j}))\|\}$, where the supremum is taken over all completely contractive homomorphisms, $\pi : \mathcal{B} \to M_k$ with k arbitrary. A dual operator algebra $\mathcal{B}$ is called* **weak*-RFD** *if this last equality holds when the completely contractive homomorphisms are also required to be weak*-continuous.*

To prove the key results in this section we require some of the definitions and the results from the paper by Paulsen [61]. We begin with the following definition.

Definition 2.4.3. *Fix a natural number k. A sequence of sets $\mathcal{S} = \{\mathcal{S}_n\}$, $\mathcal{S}_n \subseteq M_k(M_n)^+$ will be called a* matricial Schur ideal *provided that:*

(1) if (Q_{ij}), $(P_{ij}) \in \mathcal{S}_n$, then $(Q_{ij} + P_{ij}) \in \mathcal{S}_n$,

(2) if $(P_{ij}) \in \mathcal{S}_n$, and $B_1, B_2, \cdots, B_k$ are $m \times n$ matrices then $(B_i P_{ij} B_j{}^*) \in \mathcal{S}_m$.

Definition 2.4.4. *Let $\mathcal{A}$ be a k-idempotent operator algebra generated by the set of idempotents $\{E_1, E_2, \cdots, E_k\}$. Set*

$$\begin{aligned}
\mathcal{D}_n(\mathcal{A}) &= \{(W_1, W_2, \cdots, W_k) : W_i \in M_n,\ \|\sum_i W_i \otimes E_i\| \leq 1\}, \\
\mathcal{S}_n(\mathcal{A}^*\mathcal{A}) &= \{(\phi(E_i^* E_j)) : \phi : \mathcal{A}^*\mathcal{A} \to M_n \text{ is completely positive}\}, \text{ and} \\
\mathcal{S}_n(\mathcal{A}) &= \{(Q_{ij}) \in M_k(M_n)^+ : ((I - W_i^* W_j) \otimes Q_{ij}) \geq 0 \text{ for all} \\
&\qquad (W_1, \cdots, W_k) \in \mathcal{D}_m(\mathcal{A}), m \text{ arbitrary } \}.
\end{aligned}$$

It is proved in [61] that $\mathcal{S}(\mathcal{A}^*\mathcal{A}) = \{\mathcal{S}_n(\mathcal{A}^*\mathcal{A})\}$ and $\mathcal{S}(\mathcal{A}) = \{\mathcal{S}_n(\mathcal{A})\}$ are matricial Schur ideals.

Definition 2.4.5. *Let $\mathcal{A}$ be a k-idempotent operator algebra. We shall call a matricial Schur ideal $\mathcal{S} = \{\mathcal{S}_n(\mathcal{A})\}$ non-trivial if it every $Q = (Q_{ij})$ with $Q_{ij} = 0$ for $i \neq j$ and $Q_{ii} \geq 0$ is in $\mathcal{S}_n(\mathcal{A})$.*

Definition 2.4.6. *Let $\mathcal{A}$ be a k-idempotent operator algebra. We shall call a matricial Schur ideal $\mathcal{S} = \{\mathcal{S}_n(\mathcal{A})\}$ bounded if for every $Q_{ij} \in \mathcal{S}_n(\mathcal{A})$ there exists a constant $\delta > 0$ such that $(Q_{ij}) \geq \delta^2 Diag(Q_{ii})$.*

The following result is implicitly contained in [61], but the precise statement that we shall need does not appear there.

Lemma 2.4.7. *Let $\mathcal{B} = span\{F_1, \cdots, F_k\}$ be a concrete k-idempotent operator algebra. Then $\mathcal{S}(\mathcal{B}^*\mathcal{B})$ is a Schur ideal affiliated with $\mathcal{B}$, i.e., $\mathcal{B} = \mathcal{A}(\mathcal{S}(\mathcal{B}^*\mathcal{B}))$ completely isometrically.*

Proof. From Corollary 3.3 of [61] we have that the Schur ideal $\mathcal{S}(\mathcal{B}^*\mathcal{B})$ is non-trivial and bounded. Thus, we can define the algebra $\mathcal{A}(\mathcal{S}(\mathcal{B}^*\mathcal{B})) = \text{span}\{E_1, \cdots, E_k\}$, where $E_i = \sum_n \sum_{Q \in \mathcal{S}_n^{-1}(\mathcal{B}^*\mathcal{B})} \oplus Q^{1/2}(I_n \otimes E_{ii})Q^{-1/2}$ is the idempotent operator that lives on $\sum_n \sum_{Q \in \mathcal{S}_n^{-1}} \oplus M_k(M_n)$. By using Theorem 3.2 of [61] we get that $\mathcal{S}(\mathcal{A}(\mathcal{S}(\mathcal{B}^*\mathcal{B}))\mathcal{A}(\mathcal{S}(\mathcal{B}^*\mathcal{B}))^*) = \mathcal{S}(\mathcal{B}^*\mathcal{B})$ This further implies that $\mathcal{A}(\mathcal{S}(\mathcal{B}^*\mathcal{B}))\mathcal{A}(\mathcal{S}(\mathcal{B}^*\mathcal{B}))^* = \mathcal{B}^*\mathcal{B}$ completely order isomorphically under the map which sends $E_i^* E_j$ to $F_i^* F_j$. Finally, by restricting the same map to $\mathcal{A}$ we get a map which sends E_i to F_i completely isometrically. Hence, the result follows. □

Theorem 2.4.8. *Every k-idempotent operator algebra is weak*-RFD.*

Proof. Let $\mathcal{A}$ be an abstract k-idempotent operator algebra. Note that $\mathcal{A}$ is a dual operator algebra since it is a finite dimensional operator algebra. From this it follows that there exist a Hilbert space, $\mathcal{H}$ and a weak*-continuous completely isometric homomorphism, $\pi : \mathcal{A} \to B(\mathcal{H})$. Note that $\mathcal{B} = \pi(\mathcal{A})$ is a concrete k-idempotent algebra generated by the idempotents, $\mathcal{B} = \text{span}\{F_1, F_2, ..., F_k\}$ contained in $B(\mathcal{H})$. Thus, from the above lemma $\mathcal{B} = \mathcal{A}(\mathcal{S}(\mathcal{B}^*\mathcal{B}))$ completely isometrically. Further, as defined in the above lemma $\mathcal{A}(\mathcal{S}(\mathcal{B}^*\mathcal{B})) = \text{span}\{E_1, \cdots, E_k\}$, where $E_i = \sum_n \sum_{Q \in \mathcal{S}_n^{-1}(\mathcal{B}^*\mathcal{B})} \oplus Q^{1/2}(I_n \otimes E_{ii})Q^{-1/2}$ is the idempotent operator that lives on $\sum_n \sum_{Q \in \mathcal{S}_n^{-1}(\mathcal{B}^*\mathcal{B})} \oplus M_k(M_n)$..

For each $n \in \mathbb{N}$ and $Q \in \mathcal{S}_n(\mathcal{B}^*\mathcal{B})^{-1}$, we define $\pi_n^Q : \mathcal{B} \to M_k(M_n)$ via

$$\pi_n^Q(F_i) = Q^{1/2}(I_n \otimes E_{ii})Q^{-1/2}.$$

Assume for the moment that we have proven that π_n^Q is a weak*-continuous completely contractive homomorphism. Then for every $(b_{i,j}) \in M_k(\mathcal{B})$ we must have that

$$\sup_{n, Q \in \mathcal{S}_n^{-1}} \|(\pi_n^Q(b_{i,j}))\| = \|(b_{i,j})\|,$$

and hence $\|(b_{i,j})\| = \sup\{\|(\rho(b_{i,j}))\|\}$, where the supremum is taken over all weak*-continuous completely contractive homomorphisms $\rho : \mathcal{B} \to M_m$ with m arbitrary.

Since $\pi : \mathcal{A} \to \mathcal{B}$ is a complete isometry and weak*-continuous, this would imply the result for $\mathcal{A}$ by composition.

Thus, it remains to show that π_n^Q is a weak*-continuous completely contractive homomorphism on $\mathcal{B}$. Note that it is easy to check that it is a completely contractive homomorphism and it is completely isometric by the proof of the previous lemma.

Finally, the weak*-continuity of the maps π_n^Q for every n follows from the fact that $\mathcal{B}$ is finite dimensional so that the weak*-topology and the norm topology are equal. □

Theorem 2.4.9. *Every BPW complete local operator algebra of functions is weak*-RFD.*

Proof. Let $\mathcal{A}$ be a BPW complete local operator algebra of functions on the set X and let F be a finite subset of X, so that $\mathcal{A}/I_F$ is an $|F|$-idempotent operator algebra. It follows from the above lemma that $\mathcal{A}/I_F$ is weak*-RFD, i.e., for $([f_{i,j}]) \in M_k(\mathcal{A}/I_F)$ we have $\|([f_{i,j}])\| = \sup\{\|(\rho([f_{i,j}]))\|\}$ where the supremum is taken over all weak*-continuous completely contractive homomorphisms ρ from $\mathcal{A}/I_F$ into matrix algebras.

Let $(f_{i,j}) \in M_k(\mathcal{A})$, then $\|(f_{i,j})\|_{M_k(\mathcal{A})} = \sup_F \|([f_{i,j}])\|$ since $\mathcal{A}$ is local. Recall, that the weak*-topology on $\mathcal{A}$ requires all the quotient maps of the form $\pi_F : \mathcal{A} \to \mathcal{A}/I_F,\ \pi_F(f) = [f]$ to be weak*-continuous. Thus, for each finite subset $F \subseteq X$, π_F is a weak*-continuous completely contractive homomorphism. The result now follows by considering the composition of the weak*-continuous quotient maps with the weak*-continuous finite dimensional representations of each quotient algebra. □

Corollary 2.4.10. *Every local dual operator algebra of functions is weak*-RFD.*

Proof. It follows easily by using the proof of the above theorem. □

Corollary 2.4.11. *Every local operator algebra of functions is RFD.*

Proof. This result follows immediately from Therem 2.2.10 which asserts that every local operator algebra is completely isometrically contained in a BPW complete local operator algebra. □

The converse of the above may not hold true. It is easy to see that the algebra considered in Example 2.2.16, 2.2.17 serve as the counter-example for this. Both the examples were neither local nor BPW complete but one can easily see that they are RFD. Indeed, if we take $\mathcal{A} = \{P : \mathbb{D} \to \mathbb{C} : \text{ P is a polynomial}\}$ to be the algebra of polynomials equipped with the matrix normed structure such as $\|(p_{i,j})\| = \max\{\|(p_{i,j})\|_\infty, \|(p_{i,j}(2))\|\}$ then we can achieve the same norm by supping over point evaluation maps which are, in fact, one dimensional completely contractive representations. Note that,

$$\|(p_{i,j})\|_\infty = sup\{\|(E_x(p_{i,j}))\|\}$$

where $E_x : \mathcal{A} \to \mathbb{C}$ denote the usual evaluation map and the supremum is taken over all points in $X = \mathbb{D}$. Thus,

$$\|(p_{i,j})\| = \sup_{y \in X \cup \{2\}} \{\|(E_y(p_{i,j}))\|\}$$

where the supremum is taken over all point evaluation maps E_y corresponding to y which belongs to $X \cup \{2\}$. Similarly, one can work out the detail for Example 2.2.16. Thus, "local" and "BPW complete" properties of an algebra are only sufficient conditions for an operator algebra of function to be RFD.

It'll be interesting to find an example of a non-local BPW complete OPAF that is RFD. As was illustrated in the example 2.2.18 of the Section 2.2.3, constructing an example of a non-local BPW complete algebra was connected with the open problem. As of this writing, we do not have any example of such an algebra without referring to that problem. This makes us believe that it is not easy to construct an example of non-local BPW complete operator algebra that is also RFD. Our list of examples remain incomplete without an example of an algebra that is not RFD.

Chapter 3

Quantized Function Theory on Domains

3.1 Introduction

As an application of the theory of operator algebras of functions presented in the last chapter, we shall study "Function theory on Quantum domains". By "Function theory on a classical domain", we mean the study of the properties of the space of bounded analytic functions on the classical domain in $\mathbb{C}^N$. In this chapter we will develop a quantum analog of this function theory.

Whenever scalar variables are replaced by operator variables in a problem or definition, then this process is often referred to as *quantization.* It is in this sense that we would like to *quantize* the function theory on a family of complex domains. We mentioned earlier in Chapter 1 that in some sense this process has already been carried out for balls in the work of Drury [39], Popescu [68], Arveson [15], and Davidson and Pitts [35] and for polydisks in the work of Agler [2], [3], and Ball and Trent [19]. Our work is closely related to the idea of "quantizing" other domains defined by inequalities that occurs in the work of Ambrozie and

Timotin [10], Ball and Bolotnikov [17], and Kalyuzhnyi-Verbovetzkii [48], but we approach these ideas via different path of operator algebras and also the terminology is our own.

We will show that in many cases this process yields local operator algebras of functions to which the results of the earlier chapter can be applied.

We begin by defining a family of open sets for which our techniques will apply.

Definition 3.1.1. *Let $G \subseteq \mathbb{C}^N$ be an open set. If there exists a set of matrix-valued functions, $F_k = (f_{k,i,j}) : G \to M_{m_k,n_k}$, $k \in I$, whose components are analytic functions on G, and satisfy $\|F_k(z)\| < 1$, $k \in I$, then we call G an* **analytically presented domain** *and we call the set of functions $\mathcal{R} = \{F_k : G \to M_{m_k,n_k} : \ k \in I\}$ an* **analytic presentation of G.** *The subalgebra $\mathcal{A}$ of the algebra of bounded analytic functions on G generated by the component functions $\{f_{k,i,j} : 1 \leq i \leq m_k, 1 \leq j \leq n_k, k \in I\}$ and the constant function is called the* **algebra of the presentation.** *We say that $\mathcal{R}$ is a* **separating analytic presentation** *provided that the algebra $\mathcal{A}$ separates points of G.*

Remark 3.1.2. *An analytic presentation of G by a finite set of matrix-valued non-zero functions, $F_k : G \to M_{m_k,n_k}, 1 \leq k \leq K$, can always be replaced by the single block diagonal matrix-valued function, $F(z) = F_1(z) \oplus \cdots \oplus F_K(z)$ into $M_{m,n}$ with $m = m_1 + \cdots + m_K, n = n_1 + \cdots + n_K$ and we will sometimes do this to simplify proofs. But it is often convenient to think in terms of the set, especially since this will explain the sums that occur in Agler's factorization formula.*

Remark 3.1.3. *Most of the results in this chapter can be proved without the assumption that the algebra $\mathcal{A}$ generated by the component functions and constant function, separates points of G. However, we make this assumption to be consistent with the theory developed in the last chapter.*

3.2 Connection with OPAF

In the last section, we defined a notion of the algebra of the presentation $\mathcal{A}$, that possesses some of nice properties such as it is clearly an algebra of functions that separates points of the domain. As the reader can probably guess we aim at turning this algebra into an operator algebra of functions. Thus, we need to find an appropriate norm structure for $\mathcal{A}$ so that it becomes an operator algebra of functions.

Definition 3.2.1. *Let $G \subseteq \mathbb{C}^N$ be an analytically presented domain with presentation $\mathcal{R} = \{F_k = (f_{k,i,j}) : G \to M_{m_k,n_k}, k \in I\}$, let $\mathcal{A}$ be the algebra of the presentation and let $\mathcal{H}$ be a Hilbert space. A homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ is called an* **admissible representation** *provided that $\|(\pi(f_{k,i,j}))\| \leq 1$ in $M_{m_k,n_k}(B(\mathcal{H})) = B(\mathcal{H}^{n_k}, \mathcal{H}^{m_k})$, for every $k \in I$. We call the homomorphism π an* **admissible strict representation** *when these inequalities are all strictly less than 1. Given $(g_{i,j}) \in M_n(\mathcal{A})$ we set $\|(g_{i,j})\|_u = \sup\{\|(\pi(g_{i,j}))\|\}$, where the supremum is taken over all admissible representations π of $\mathcal{A}$. We let $\|(g_{i,j})\|_{u_0}$ denote the supremum that is obtained when we restrict to admissible strict representations.*

The theory of [10] and [17] studies domains defined as above with the additional restrictions that the set of defining functions is a finite set of polynomials. However, they do not need their polynomials to separate points, while we shall shortly assume that our presentations are separating, in order to invoke the results of the previous sections. This latter assumption can, generally, be dropped in our theory, but it requires some additional argument. There are several other places where our results and definitions given below differ from theirs. So while our results extend their results in many cases, in other cases we are using different definitions and direct comparisons of the results are not so clear.

Proposition 3.2.2. *Let G have a separating analytical presentation and let $\mathcal{A}$ be the* *algebra of the presentation. Then $\mathcal{A}$ endowed with either of the family of norms $\|\cdot\|_u$ or $\|\cdot\|_{u_0}$ is an operator algebra of functions on G.*

Proof. It is clear that it is an operator algebra and by definition it is an algebra of functions on G. It follows from the hypotheses that it separates points of G. Finally, for every $\lambda = (\lambda_1, ..., \lambda_N) \in G$, we have a representation of $\mathcal{A}$ on the one-dimensional Hilbert space given by $\pi_\lambda(f) = f(\lambda)$. Hence, $|f(\lambda)| \leq \|f\|_u$ and so $\mathcal{A}$ is an operator algebra of functions on G. $\square$

For the rest of the discussion, we may denote the operator algebra $(\mathcal{A}, \|.\|_u)$ by $\mathcal{A}$ and $(\mathcal{A}, \|.\|_{u_0})$ by $\mathcal{A}_0$. Note that we only have $\mathcal{A} \subseteq_{cc} \mathcal{A}_0$, even though $\mathcal{A} = \mathcal{A}_0$ as sets.

3.3 GNFT and GNPP

In this section, we obtain a generalized Nevanlinna factorization theorem and a generalized solution to the Nevanlinna Pick problem for the algebra of the presentation. The results in this section form a building block of our strategy to be able to prove results similar to the ones obtained by Ball-Bolotnikov, details of which can be found in Section 1.4. To prove results in this section, we use an important tool from the theory of operator algebras, that is, the BRS theorem.

First, we give some necessary terminology. It will be convenient to say that matrices, $A_1, \ldots A_m$ are of *compatible sizes* if the product, $A_1 \cdots A_m$ exists, that is, provided that each A_i is an $n_i \times n_{i+1}$ matrix.

Given an analytically presented domain G, with an analytic presentation $\mathcal{R} = \{F_k : k \in I\}$. Let F_1 denote the constant function 1. By an **admissible block diagonal**

matrix over G we mean a block diagonal matrix-valued function of the form $D(z) = diag(F_{k_1}, ..., F_{k_m})$ where $k_i \in I \cup \{1\}$ for $1 \leq i \leq m$. Thus, we are allowing blocks of 1's in $D(z)$. Finally, given a matrix B we let $B^{(q)} = diag(B, ..., B)$ denote the block diagonal matrix that repeats B q times.

Theorem 3.3.1. *Let G be an analytically presented domain with presentation $\mathcal{R} = \{F_k = (f_{k,i,j}) : G \to M_{m_k,n_k}, k \in I\}$, let $\mathcal{A}$ be the algebra of the presentation and let $P = (p_{ij}) \in M_{m,n}(\mathcal{A})$, where m, n are arbitrary. Then the following are equivalent:*

(i) $\|P\|_u < 1$,

(ii) *there exist an integer l, matrices of scalars C_j, $1 \leq j \leq l$ with $\|C_j\| < 1$, and admissible block diagonal matrices $D_j(z), 1 \leq j \leq l$, which are of compatible sizes and are such that*

$$P(z) = C_1 D_1(z) \cdots C_l D_l(z),$$

(iii) *there exist a positive, invertible matrix $R \in M_m$, and matrices $P_0, P_k \in M_{m,r_k}(\mathcal{A}), k \in K$, where $K \subseteq I$ is a finite set, such that*

$$I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k \in K} P_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)} P_k(w)^*$$

where $r_k = q_k m_k$ and $z = (z_1, ..., z_N)$, $w = (w_1, ..., w_N) \in G$.

Proof. Although we will not logically need it, we first show that (ii) implies (i), since this is the easiest implication and helps to illustrate some ideas. Note that if $\pi : \mathcal{A} \to B(\mathcal{H})$ is any admissible representation, then the norm of π of any admissible block diagonal matrix is at most 1. Thus, if P has the form of (ii), then for any admissible π, we will have $(\pi(p_{i,j}))$ expressed as a product of scalar matrices and operator matrices all of norm at most one and hence, $\|(\pi(P_{i,j}))\| \leq \|C_1\| \cdots \|C_l\| < 1$. Thus, $\|P\|_u \leq \|C_1\| \cdots \|C_l\| < 1$.

We now prove that (i) implies (ii). The ideas of the proof are similar to [62, Corollary 18.2], [24, Corollary 2.11] and [51, Theorem 1] and use in an essential way the abstract characterization of operator algebras. For each $m, n \in \mathbb{N}$, one proves that $\|P\|_{m,n} := inf\{\|C_1\| \dots \|C_l\|\}$, defines a norm on $M_{m,n}(\mathcal{A})$, where the infimum is taken over all l and all ways to factor $P(z) = C_1 D_1(z_{i_1}) \cdots C_l D_l(z_{i_l})$ as a product of matrices of compatible sizes with scalar matrices $C_j, 1 \leq j \leq l$ and admissible block diagonal matrices $D_j, 1 \leq j \leq l$.

Moreover, one can verify that $\mathcal{M}_{m,n}(\mathcal{A})$ with this family $\{\|.\|_{m,n}\}_{m,n}$ of norms satisfies the BRS axioms for an abstract unital operator algebra mentioned in the Chapter 2 and hence by the Blecher-Ruan-Sinclair representation theorem [25](see also [62]) there exists a Hilbert space $\mathcal{H}$ and a unital completely isometric isomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$.
Thus, for every $m, n \in \mathbb{N}$ and for every $P = (p_{ij}) \in \mathcal{M}_{m,n}(\mathcal{A})$, we have that $\|P\|_{m,n} = \|(\pi(p_{ij}))\|$. However, $\|\pi^{(m_k,n_k)}(F_k)\| = \|(\pi(f_{k,i,j}))\| \leq 1$ for $1 \leq i \leq K$, and so, π is an admissible representation. Thus, $\|P\|_{m,n} = \|(\pi(p_{ij}))\| \leq \|P\|_u$. Hence, if $\|P\|_u < 1$, then $\|P\|_{m,n} < 1$ which implies that such a factorization exists. This completes the proof that (i) implies (ii).

We will now prove that (ii) implies (iii). Suppose that P has a factorization as in (ii). Let $K \subseteq I$ be the finite subset of all indices that appear in the block-diagonal matrices appearing in the factorization of P. We will use induction on l to prove that (iii) holds.

First, assume that $l = 1$ so that $P(z) = C_1 D_1(z)$. Then,

$$\begin{aligned} I_m - P(z)P(w)^* &= I_m - (C_1 D_1(z))(C_1 D_1(w))^* \\ &= (I_m - C_1 C_1^*) + C_1 \left(I - D_1(z) D_1(w)^* \right) C_1^*. \end{aligned}$$

Since $D_1(z)$ is an admissible block diagonal matrix the (i, i)-th block diagonal entry of $I - D_1(z)D_1(w)^*$ is $I - F_{k_i}(z)F_{k_i}(w)^*$ for some finite collection, k_i.

Let E_k be the diagonal matrix that has 1's wherever F_k appears(so $E_k = 0$ when there is no F_k term in D_1). Hence,

$$C_1(I - D_1(z)D_1(w)^*)C_1^* = \sum_k C_1 E_k (I - F_k(z)F_k(w)^*)E_k C_1^*.$$

Therefore, gathering terms for common values of i,

$$I_m - P(z)P(w)^* = R_0 + \sum_{k \in K} P_k (I - F_k(z)F_k(w)^*)P_k^*,$$

where $R_0 = I_m - C_1C_1^*$ is a positive, invertible matrix and P_i is, in this case a constant. Thus, the form (iii) holds, when $l = 1$.

We now assume that the form (iii) holds for any $R(z)$ that has a factorization of length at most $l-1$, and assume that

$$P(z) = C_1D_1(z)\cdots D_{l-1}(z)C_lD_l(z) = C_1D_1(z)R(z),$$

where $R(z)$ has a factorization of length $l-1$.

Note that a sum of expressions such as on the right hand side of (iii) is again such an expression. This follows by using the fact that given any two expressions $A(z),\ B(z)$, we can write

$$A(z)A(w)^* + B(z)B(w)^* = C(z)C(w)^*,$$

where $C(z) = (A(z),\ B(z))$.

Thus, it will be sufficient to show that $I_m - P(z)P(w)^*$ is a sum of expressions as above. To this end we have that,

$$I_m - P(z)P(w)^* = (I_m - C_1D_1(z)D_1(w)^*C_1^*) + (C_1D_1(z))(I - R(z)R(w)^*)(D_1(w)^*C_1^*).$$

The first term of the above equation is of the form as on the right hand side of (iii) by case $l = 1$. Also, the quantity $(I - R(z)R(w)^*) = R_0R_0^* + R_0(z)R_0(w)^* + \sum_{k\in K} R_k(z)(I -$

$F_k(z)F_k(w)^*)^{(q_k)}R_k(w)^*$ by the inductive hypothesis. Hence,

$$C_1D_1(z)(I - R(z)R(w)^*)D_1(w)^*C_1^* =$$
$$(C_1D_1(z)R_0)(C_1D(w)R_0)^* + [C_1D_1(z)R_0(z)][C_1D_1(w)R_0(w)]^* +$$
$$\sum_{k\in K}[C_1D_1(z)R_k(z)](I - F_k(z)F_k(w)^*)^{(q_k)}[C_1D_1(w)R_k(w)]^*.$$

Thus, we have expressed $(I - P(z)P(w)^*)$ as a sum of two terms both of which can be written in the form desired. Using again our remark that the sum of two such expressions is again such an expression, we have the required form.

Finally, we will prove (iii) implies (i). Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be an admissible representation and let $P = (p_{i,j}) \in M_{m,n}(\mathcal{A})$ have a factorization as in (iii). To avoid far too many superscripts we simplify $\pi^{(m,n)}$ to Π.

Now observe that

$$I_m - \Pi(P)\Pi(P)^* = \Pi(R) + \Pi(P_0)\Pi(P_0)^* + \sum_{k\in K}\Pi(P_k)(I - \Pi(F_k)\Pi(F_k)^*)^{(q_k)}(\Pi(P_k))^*.$$

Clearly the first two terms of the sum are positive. But since π is an admissible representation, $\|\Pi(F_k)\| \le 1$ and hence, $(I - \Pi(F_k)\Pi(F_k)^*) \ge 0$. Hence, each term on the right hand side of the above inequality is positive and since R is strictly positive, say $R \ge \delta I_m$ for some scalar $\delta > 0$, we have that $I_m - \Pi(P)\Pi(P)^* \ge \delta I_m$.
Therefore, $\|\Pi(P)\| \le \sqrt{1-\delta}$. Thus, since π was an arbitrary admissible representation, $\|P\|_u \le \sqrt{1-\delta} < 1$, which proves (i). □

When we require the functions in the presentation to be row vector-valued, then the above theory simplifies somewhat and begins to look more familiar. Let G be an analytically presented domain with presentation $F_k : G \to M_{1,n_k},\ k \in I$. We identify $M_{1,n}$ with the

Hilbert space $\mathbb{C}^n$ so that $1 - F_k(z)F_k(w)^* = 1 - \langle F_k(z), F_k(w)\rangle$, where the inner product is in $\mathbb{C}^n$. In this case we shall say that G is *presented by vector-valued functions.*

Corollary 3.3.2. *Let G be presented by vector-valued functions, $F_k = (f_{k,j}) : G \to M_{1,n_k}, k \in I$, let $\mathcal{A}$ be the algebra of the presentation and let $P = (p_{ij}) \in M_{m,n}(\mathcal{A})$. Then the following are equivalent:*

(i) $\|P\|_u < 1$,

(ii) there exist an integer l, matrices of scalars C_j, $1 \le j \le l$ with $\|C_j\| < 1$, and admissible block diagonal matrices $D_j(z), 1 \le j \le l$, which are of compatible sizes and are such that

$$P(z) = C_1 D_1(z) \cdots C_l D_l(z),$$

(iii) there exist a positive, invertible matrix $R \in M_m$, and matrices $P_0 \in M_{m,r_0}(\mathcal{A})$, $P_k \in M_{m,r_k}(\mathcal{A}), k \in K$, where $K \subseteq I$ is finite, such that

$$I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k\in K}(1 - \langle F_k(z), F_k(w)\rangle)P_k(z)P_k(w)^*$$

where $z = (z_1, ..., z_N)$, $w = (w_1, ..., w_N) \in G$.

Corollary 3.3.3. *Let Y be a subset of an analytically presented domain G with analytic presentation $F_k = (f_{k,i,j}) : G^- \to M_{m_k,n_k}, 1 \le k \le K$ and let $\pi_Y : \mathcal{A} \to \mathcal{A}/I_Y$ be the quotient map, where $I_Y = \{f \in \mathcal{A} : f|_Y = 0\}$. Let $\mathcal{A}$ be the algebra of the presentation and let $P = (p_{ij}) \in M_{m,n}(\mathcal{A})$, where m, n are arbitrary. Then the following are equivalent:*

(i) $\|(\pi_Y(p_{ij}))\|_u < 1$,

(ii) there exist an integer l, matrices of scalars C_j, $1 \le j \le l$ with $\|C_j\| < 1$, and admissible block diagonal matrices $D_j(z), 1 \le j \le l$, which are of compatible sizes and are such that $P(z) = C_1 D_1(z) \cdots C_l D_l(z)$ for every $z \in Y$,

(iii) there exist a positive, invertible matrix $R \in M_m$*, and matrices* $P_k \in M_{m,r_k}(\mathcal{A}), 0 \leq k \leq K$*, such that*

$$I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k=1}^{K} P_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)} P_k(w)^*$$

where $r_k = q_k m_k$ *and* $z = (z_1, ..., z_N),\ w = (w_1, ..., w_N) \in S$.

Proof. To show (i) implies (ii) we assume that $\|(\pi_Y(p_{ij}))\|_u < 1$. Then there exists $q_{ij} \in I_Y$ such that $\|(p_{ij} + q_{ij})\|_u < 1$ and hence by using theorem 3.3.1 we get (ii). For the converse, assume that there exists an integer l, matrices of scalars $C_j,\ 1 \leq j \leq l$ with $\|C_j\| < 1$ and admissible block diagonal matrices $D_j(z),\ 1 \leq j \leq l$, which are of compatible sizes and are such that $P(z) = C_1 D_1(z) \cdots C_l D_l(z)$ for every $z \in S$. Let $(r_{ij}(z)) = C_1 D_1(z) \cdots C_l D_l(z)$ then $\|(r_{ij})\|_u < 1$ and $r_{ij}(\lambda) = p_{ij}(\lambda)\ \forall\ \lambda \in Y$. This shows that $\|(\pi_Y(p_{ij}))\|_u \leq \|(p_{ij}) + (r_{ij} - p_{ij})\|_u = \|(r_{ij})\|_u < 1$. □

Remark 3.3.4. *It can be proved that (ii) in the above corollary implies that the factorization formula given in (iii) of Theorem 3.3.1 hold true for all the points in the set* S *by the same proof as given in the Theorem 3.3.1.*

The following result gives us a Nevanlinna-type result for the algebra of presentation.

Theorem 3.3.5. *Let* Y *be a finite subset of an analytically presented domain* G *with separating analytic presentation* $F_k = (f_{k,i,j}) : G \to M_{m_k,n_k}, k \in I$*, let* $\mathcal{A}$ *be the algebra of the presentation and let* P *be a* $M_{m,n}$*-valued function defined on a finite subset* $Y = \{x_1, \cdots, x_l\}$ *of* G*. Then the following are equivalent:*

(i) there exists $\tilde{P} \in M_{mn}(\mathcal{A})$ *such that* $\tilde{P}|_Y = P$ *and* $\|\tilde{P}\|_u < 1$*,*

(ii) there exist a positive, invertible matrix $R \in M_m$, *and matrices* $P_0 \in M_{m,r_0}(\mathcal{A})$, $P_k \in M_{m,r_k}(\mathcal{A}), k \in K$, *where* $K \subseteq I$ *is a finite set, such that*

$$I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k \in K} P_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)} P_k(w)^*,$$

where $r_k = q_k m_k$ *and* $z = (z_1, ..., z_N),\ w = (w_1, ..., w_N) \in Y$.

Proof. Note that $(i) \Rightarrow (ii)$ follows immediately as a corollary of Theorem 3.3.1. Thus it only remains to show that $(ii) \Rightarrow (i)$. Since $\mathcal{A}$ is an operator algebra of functions, therefore, $\mathcal{A}/I_Y$ is a finite dimensional operator algebra of idempotents and $\mathcal{A}/I_Y = \text{span}\{E_1, \cdots, E_l\}$ where $l = |Y|$. Thus there exists a Hilbert space H_Y and a completely isometric representation π of $\mathcal{A}/I_Y$. By Theorem 2.3.5, there exists a kernel K_Y such that $\pi(\mathcal{A}/I_Y) = \mathcal{M}(K_Y)$ completely isometrically under the map $\rho : \pi(\mathcal{A}/I_Y) \to \mathcal{M}(K_Y)$ which sends $\pi(B)$ to M_f, where $B = \sum_{i=1}^{l} a_i \pi(E_i)$ and $f : Y \to \mathbb{C}$ is a function defined by $f(x_i) = a_i$. Note that

$$((I - F_k(x_i)F_k(x_j)) \otimes K_Y(x_i, x_j)) = \\ ((I - \pi(F_k + I_Y)(x_i)\pi(F_k + I_Y)(x_j)^*) \otimes K_Y(x_i, x_j))_{ij} \geq 0,$$

since $\|\pi(F_k + I_Y)\| \leq \|F_k\|_u \leq 1$ for all $k \in I$. From this it follows that $((I_m - (\pi(P + I_Y))(x_i)(\pi(P+I_Y))(x_j)^*) \otimes K_Y(x_i, x_j))_{ij} \geq 0$. Using that $R > 0$, we get that $\|\pi(P+I_Y)\| < 1$. This shows that there exists $\tilde{P} \in \mathcal{A}$ such that $\tilde{P}|_Y = P$ and $\|\tilde{P}\|_u < 1$. This completes the proof. □

Before we close this section, it would be valuable to make a note that the results in this section can be proved for a non-commutative algebra as well. The proofs of all the results in this section hold good for any algebra that is generated by any set of free generators.

3.4 Main Result

In this section, we do justice to the title of this chapter and prove the main result of this chapter. In a true sense, we study "quantized function theory" in this section. First, we define quantized version of an analytically presented domain. Second, we aim at establishing a relation between the algebra of the presentation and the bounded analytic functions on these domains which is achieved by our main result. In the main result, we obtain a "neat" connection between the two algebras which allows us to pull back results from the previous chapter. Also, with the help of this connection and the factorization results obtained in the earlier section for the algebra of the presentation, we are able to prove GNFT for the algebra of bounded analytic functions on quantum domains.

First, we turn towards defining quantized versions of the bounded analytic functions on these domains. For this we need to recall that the joint Taylor spectrum [77] of a commuting N-tuple of operators $T = (T_1, ..., T_N)$, is a compact set, $\sigma(T) \subseteq \mathbb{C}^N$ and that there is an analytic functional calculus [78], [79] defined for any function that is holomorphic in a neighborhood of $\sigma(T)$.

Definition 3.4.1. *Let $G \subseteq \mathbb{C}^N$ be an analytically presented domain, with presentation $\mathcal{R} = \{F_k : G^- \to M_{m_k,n_k}, k \in I\}$. We define the* **quantized version of G** *to be the collection of all commuting N-tuples of operators,*

$$\mathcal{Q}(G) = \{T = (T_1, T_2, \dots, T_N) \in B(\mathcal{H}) : \sigma(T) \subseteq G \text{ and } \|F_k(T)\| \leq 1, \ \forall \ k \in I\},$$

where $\mathcal{H}$ is an arbitrary Hilbert space. We set

$$\mathcal{Q}_{0,0}(G) = \{T = (T_1, T_2, \dots, T_N) \in M_n : \sigma(T) \subseteq G \text{ and } \|F_k(T)\| \leq 1, \ \forall \ k \in I\},$$

where n is an arbitrary positive integer.

Note that if we identify a point $(\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ with an N-tuple of commuting operators on a one-dimensional Hilbert space, then we have that $G \subseteq \mathcal{Q}(G)$.

If $T = (T_1, ..., T_N) \in \mathcal{Q}(G)$, is a commuting N-tuple of operators on the Hilbert space $\mathcal{H}$, then since the joint Taylor spectrum of T is contained in G, we have that if f is analytic on G, then there is an operator $f(T)$ defined and the map $\pi : Hol(G) \to B(\mathcal{H})$ is a unital continuous homomorphism, where $Hol(G)$ denotes the algebra of analytic functions on G [79].

Definition 3.4.2. *Let $G \subseteq \mathbb{C}^N$ be an analytically presented domain, with presentation $\mathcal{R} = \{F_k : G^- \to M_{m_k,n_k},\ k \in I\}$. We define $H^\infty_\mathcal{R}(G)$ to be the set of functions $f \in Hol(G)$, such that $\|f\|_\mathcal{R} \equiv \sup\{\|f(T)\| : T \in \mathcal{Q}(G)\}$ is finite. Given $(f_{i,j}) \in M_n(H^\infty_\mathcal{R}(G))$, we set $\|(f_{i,j})\|_\mathcal{R} = \sup\{\|(f_{i,j}(T))\| : T \in \mathcal{Q}(G)\}$.*

We are interested in determining a connection between the algebra of the presentation and $H^\infty_\mathcal{R}(G)$ and whether or not the $\|\cdot\|_\mathcal{R}$ norm is attained on the smaller set $\mathcal{Q}_{0,0}(G)$. Note that since each point in $G \subseteq \mathcal{Q}(G)$, we have that $H^\infty_\mathcal{R}(G) \subseteq H^\infty(G)$, and $\|f\|_\infty \leq \|f\|_\mathcal{R}$. Also, we have that $\mathcal{A} \subseteq H^\infty_\mathcal{R}(G)$ and for $(f_{i,j}) \in M_n(\mathcal{A})$, $\|(f_{i,j})\|_\mathcal{R} \leq \|(f_{i,j})\|_u$. Indeed, it follows from the fact that every $T \in \mathcal{Q}(G)$ gives rise to an admissible representation of $H^\infty_\mathcal{R}(G)$. The inclusion of $\mathcal{A}$ into $H^\infty_\mathcal{R}(G)$ might not even be isometric.

Recall, $\mathcal{A}_0 = (\mathcal{A}, \|.\|_{u_0})$ where $\|.\|_{u_0}$ is a norm on $\mathcal{A}$ obtained by supping over all strict admissible representations on $\mathcal{A}$. It is natural to wonder about the connection of the algebra $\mathcal{A}_0$ with $H^\infty_\mathcal{R}(G)$. We know that $\mathcal{A} \subseteq_{cc} \mathcal{A}_0$ and $\mathcal{A} = \mathcal{A}_0$ as sets, still the connection between $\mathcal{A}_0$ and $H^\infty_\mathcal{R}(G)$ is not at all obvious.

Lemma 3.4.3. *Let $\pi : H^\infty_R(G) \to \mathbb{C}$ be completely contractive and weak*-continuous homomorphism. If $F = (f_{i,j}) \in \mathcal{R}$, then $\|(\pi(f_{i,j}))\| < 1$.*

Proof. We know that $\|(\pi(f_{i,j}))\| \leq 1$. Suppose that it has norm equal to 1, then since it is a finite matrix, there are unit vectors v, w such that $\langle(\pi(f_{i,j}))v, w\rangle = 1$. Let $h(z) = \langle(f_{i,j}(z))v, w\rangle$, then $\pi(h) = 1$.

Since $\|F(z)\| < 1$ for each $z \in G$, we have that $|h(z)| < 1$ for each $z \in G$. Hence, $h(z)^k \to 0$ for each z, which further yields that $h^k \to 0$ in the weak*-topology, which implies that $\pi(h)^k \to 0$. This contradicts $\pi(h) = 1$. □

We would like to record a fact from matrix theory that for every abelian subalgebra $\mathcal{B}$ of M_n there exists an unitary $U \in M_n$ such that $U\mathcal{B}U^* \subseteq T_n$, where T_n is the space of upper triangular matrices, see [46]. Thus, in particular, if $\pi : H^\infty_R(G) \to M_n$ is a completely contractive and weak*-continuous homomorphism then there exists an unitary U such that $\sigma : H^\infty_R(G) \to T_n$ defined via $\sigma(.) = U\pi(.)U^*$ is also a completely contractive and weak*-continuous homomorphism.

Lemma 3.4.4. *Let $\sigma : H^\infty_R(G) \to T_n$ be a completely contractive and weak*-continuous homomorphism. If $F = (f_{i,j}) \in \mathcal{R}$, then $|\langle(\sigma(f_{i,j}))e_k, e_k\rangle| < 1$ for all k.*

Proof. We define

$$\delta : H^\infty_R(G) \to \mathbb{C} \quad via \quad \delta(.) = \langle\sigma(.)e_i, e_i\rangle.$$

Then it follows that δ is a homomorphism from the hypothesis that σ is a homomorphism. Indeed, if we let $f, g \in H^\infty_R(G)$ then

$$\begin{aligned}\sigma(fg) = \langle\pi(f)\pi(g)U^*e_i, U^*e_i\rangle &\overset{homo.}{=} \langle U\pi(f)U^*U\pi(g)U^*e_i, e_i\rangle \\ &\overset{*}{=} \langle U\pi(f)U^*e_i, e_i\rangle\langle U\pi(g)U^*e_i, e_i\rangle = \sigma(f)\sigma(g).\end{aligned}$$

The equality (*) follows from the fact that the diagonal entry of the product of two triangular matrices is equal to the product of the diagonal entries of the two triangular matrices.

It follows from a straightforward argument that δ is also a completely contractive and weak*-continuous map. Finally, the result follows by using Lemma 3.4.3. $\square$

Lemma 3.4.5. *Let $\sigma : H_R^\infty(G) \to T_n$ be a completely contractive and weak*-continuous homomorphism. If $F = (f_{i,j}) \in \mathcal{R}$, then there exists a completely contractive homomorphism $\sigma_z : H_R^\infty(G) \to T_n$ such that $\|(\sigma_z(f_{ij}))\| < 1$ for every $z \in \mathbb{D}$.*

Proof. For a fixed $z \in \mathbb{D}$ and a triangular matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & t_{nn} \end{bmatrix}$$

we define a triangular matrix $T_z = V_z T V_z^*$ where $V_z = diag(1, z, z^2, \cdots, z^{n-1})$. Thus,

$$T_z = \begin{bmatrix} t_{11} & zt_{12} & \dots & z^{n-1}t_{1n} \\ 0 & t_{22} & \dots & z^{n-2}t_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & t_{nn} \end{bmatrix}$$

Using this operation, we define the desired map $\sigma_z : H_R^\infty(G) \to T_n$ as $\sigma_z(f) = (\sigma(f))_z$. It is not so hard to see that σ_z is a homomorphism by brute force calculations. To avoid calculations, one can even adopt a slicker way of proving this by invoking the maximum modulus theorem. Note that σ_z is an analytic function of z on the unit disc $\mathbb{D}$ and on the unit circle V_z turns out to be a unitary matrix. For a fixed $f, g \in H_R^\infty(G)$, it is is obvious that

$$\sigma_{e^{i\theta}}(fg) = \sigma_{e^{i\theta}}(f)\sigma_{e^{i\theta}}(g).$$

If we define a linear functional $L : \mathbb{D} \to \mathbb{C}$ for a fixed pair of $\alpha, \beta \in \mathbb{C}^n$ with $\|\alpha\| = 1,\ \|\beta\| = 1$ via $L_{\alpha,\beta}(z) = \langle(\sigma_z(fg) - \sigma_z(f)\sigma_z(g))\alpha, \beta\rangle$, then L is a complex-valued analytic function on $\mathbb{D}$ and $L_{\alpha,\beta}(e^{i\theta}) = 0$ for every $\theta \in [0, 2\pi]$. Thus, by the maximum modulus theorem we have that $L_{\alpha,\beta}(z) = 0$ for every $\alpha,\ \beta \in C^n$ and $z \in \mathbb{D}$. This proves that $\sigma_z(gh) = \sigma_z(g)\sigma_z(h)$ for every $g,\ h \in H^\infty_R(G)$ and $z \in \mathbb{D}$.

To prove the final conclusion, note that by the maximum modulus theorem for a fixed $z \in D$, we have that

$$|\langle\sigma_z(g)\alpha, \beta\rangle| \leq |\langle\sigma_{e^{i\theta}}(g)\alpha, \beta\rangle| \leq |\langle U_{e^{i\theta}}\sigma(g)U^*_{e^{i\theta}}\alpha, \beta\rangle| \leq \|\sigma(g)\|.$$

By taking the supremum over all $\alpha,\ \beta \in \mathbb{C}^n,\ \|\alpha\| = \|\beta\| = 1$ and $z \in \mathbb{D}$, we get that

$$\sup_{z\in\mathbb{D}} \|\sigma_z(g)\| \leq \|\sigma(g)\|$$

for every $g \in H^\infty_{\mathcal{R}}(\mathbb{D})$. Similar arguments can be used to prove that $\sup_{z\in\mathbb{D}} \|(\sigma_z(g_{ij}))\| \leq \|(\sigma(g_{ij}))\|$. This together with the fact that $\|(\sigma(f_{ij}))\| \leq 1$ implies that $\|(\sigma_z(f_{ij}))\| \leq 1$ for every $F = (f_{ij}) \in \mathcal{R}$ and $z \in \mathbb{D}$. Also, note that $\|(\sigma_0(f_{ij}))\| = \max_k |\langle(\pi(f_{ij})e_k, e_k\rangle| < 1$, because of the previous lemma. Finally, if there exists $z_0 \in \mathbb{D}$ such that the conclusion fails, that is, $\|(\sigma_{z_0}(f_{ij}))\| = 1$. Then by a compactness argument, there exist $k_1,\ k_2$ such that

$$|\langle(\sigma_{z_0}(f_{ij}))k_1, k_2\rangle| = 1 \geq |\langle(\sigma_z(f_{ij}))k_1, k_2\rangle|$$

for every $z \in \mathbb{D}$. Thus, by the maximum modulus theorem we have that $|\langle(\sigma_z(f_{ij}))k_1, k_2\rangle| = 1$ for every $z \in \mathbb{D}$ which contradicts the fact $\|(\sigma_0(f_{ij}))\| < 1$. □

In summary, the above series of lemmas proves that for every finite dimensional completely contractive weak*-continuous homomorphism $\pi : H^\infty_{\mathcal{R}}(G) \to M_n$ there exists a set of strict admissible representations $\sigma_z : H^\infty_{\mathcal{R}}(G) \to T_n$ and $\|(\sigma_z(g_{ij}))\| \leq \|(\sigma(g_{ij}))\| \leq \|(\pi(g_{ij}))\|$ for every $g_{ij} \in H^\infty_{\mathcal{R}}(G)$.

We are now in a position to prove a theorem that establishes a connection between the algebra of the presentation of the domain G equipped with $\|.\|_{u_0}$ norm and $H^\infty_{\mathcal{R}}(G)$.

Theorem 3.4.6. *Let G be an analytically presented domain and let $\mathcal{A}$ be the algebra of the presentation. Then $\|H\|_{\mathcal{R}} \leq \|H\|_{u_0}$ for every $H \in M_n(\mathcal{A})$ and for all n.*

Proof. Let $\pi : H^\infty_{\mathcal{R}}(G) \to M_n$ be a completely contractive weak*-continuous homomorphism. Then, for every $z \in \mathbb{D}$, there exists a homomorphism $\sigma_z : H^\infty_{\mathcal{R}}(G) \to T_n$ such that $\|(\sigma_z(f_{ij}))\| < 1$ and $\|(\sigma_z(h_{ij}))\| \leq \|(\pi(h_{ij}))\|$ for every $(h_{ij}) \in M_n(H^\infty_{\mathcal{R}}(G))$.

We claim that $\sup_{z\in\mathbb{D}} \|(\sigma_z(h_{ij}))\| \geq \|(\pi(h_{ij}))\|$. To prove this claim, we need to recall that $\sigma_z = V_z U \pi U^* V_z$ where $V_z = diag(1, z, \cdots, z^{n-1})$ and U is the unitary matrix in M_n. Clearly,

$$|\langle(\pi(h_{ij}))e_j, e_i\rangle| = \lim_{r\nearrow 1} |\langle(\sigma_{re^{i\theta}}(h_{ij}))U^* e_j, U^* e_i\rangle|.$$

Note that the above equality holds true for any two column vectors h, k in an appropriate finite dimensional space. Let us assume that $\|h\| = \|k\| = 1$. Thus,

$$|\langle(\pi(h_{ij}))h, k\rangle| = \lim_{r\nearrow 1} |\langle(\sigma_{re^{i\theta}}(h_{ij}))U^* h, U^* k\rangle| \leq \limsup_{r\nearrow 1} \|(\sigma_{re^{i\theta}}(h_{ij}))\| \leq \sup_{z\in\mathbb{D}} \|(\sigma_z(h_{ij}))\|.$$

By taking the supremum over all h, k with $\|h\| = \|k\| = 1$, we obtain our claim. This further implies that $\|(pi(h_{ij}))\| \leq \|(h_{ij})\|_{u_0}$. Finally, by Theorem 3.4.11, it is enough to take the supremum over all completely contractive weak*-continuous homomorphisms to obtain $\|f\|_{\mathcal{R}}$ and hence our result. □

We shall see in the example section that for most of the algebras $\mathcal{A}$, we can prove that $\mathcal{A} \cong \mathcal{A}_0$ completely isometrically, still we do not know this in general.

Corollary 3.4.7. *Let G be an analytically presented domain and let $\mathcal{A}$ be the algebra of the presentation. If $\mathcal{A}$ is a local operator algebra of functions, then $\mathcal{A} =_{ci} \mathcal{A}_0$, that is, $\mathcal{A} = \mathcal{A}_0$*

completely isometrically.

Remark 3.4.8. *If we look closely then we can prove that $\mathcal{A} =_{ci} \mathcal{A}_0$ under a weaker assumption that the algebra is RFD by constructing a finite dimensional strict admissible representation using a finite dimensional completely contractive representation as done in Lemma 3.4.3. However, we believe that the converse of this may not hold true. Though, we do not have any counter-example to support our belief. Note that to be able to establish this, we would need an example of a non-RFD algebra and we remarked earlier in Chapter 2 that problem of finding an example of non-RFD algebra is still open.*

Recall, the norm on the BPW completion of an operator algebra of functions depends on the norm of the algebra. Thus, for each norm on $\mathcal{A}$ we get a norm structure on $\widetilde{\mathcal{A}}$. It is natural to wonder if anything can be said about the BPW completion of these algebras: $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{A}_0}$.

Corollary 3.4.9. *Let G be an analytically presented domain and let $\mathcal{A}$ be the algebra of the presentation. Then $\widetilde{\mathcal{A}} =_{ci} \widetilde{\mathcal{A}_0}$.*

Proof. Recall, if $f \in \widetilde{\mathcal{A}}$ then $\|f\| = \inf\{C : \exists\ f_\lambda \in \mathcal{A},\ f_\lambda \to f,\ \|f_\lambda\|_u \leq C\}$. We may denote the norm on the BPW completion of $\mathcal{A}_0$ by $\|.\|_0$. It is obvious that $\|f\|_{u_0} \leq \|f\|_u$ for every $f \in \mathcal{A}$ and consequently, it follows that for every $g \in \widetilde{\mathcal{A}}$ we have that $\|g\|_0 \leq \|g\|$. Suppose that $g \in \widetilde{\mathcal{A}_0}$ with $\|g\|_0 \leq 1$. Thus, there exists a sequence $\{g_\lambda\}$ of functions in $\mathcal{A}$ such that $g_\lambda \to g$ pointwise and $\|g_\lambda\|_{u_0} \leq 1$.

For a given $\epsilon > 0$ and a fixed $x \in G$, there exists λ_0 such that $|g_{\lambda_0}(x) - g(x)| < \epsilon$. From the above theorem, we get $\|g_\lambda\|_L \leq 1$ which further implies that there exists a net $\{g^F_{\lambda_0}\} \in \mathcal{A}$ such that $\|g^F_{\lambda_0}\|_u \leq 1$ and $|g^F_{\lambda_0}(x) - g(x)| \to 0$. This shows that $\|g\|_u \leq 1$. On the similar lines, we can prove this for matrices. With this we conclude, $\widetilde{\mathcal{A}}$ is completely isometrically isomorphic to $\widetilde{\mathcal{A}_0}$. $\square$

Corollary 3.4.10. *Let G be an analytically presented domain and let $\mathcal{A}$ be the algebra of the presentation. If $\mathcal{A} =_{ci} \tilde{\mathcal{A}}$, then $\mathcal{A}_0 =_{ci} \widetilde{\mathcal{A}_0}$.*

Proof. The proof is immediate from the above corollary. $\square$

We are now in a position to state and prove our main result which establishes a very precise connection between the algebra of the presentation ($\mathcal{A}_0$ or $\mathcal{A}$) and $H^\infty_{\mathcal{R}}(G)$. The algebra of the bounded analytic function on a quantum domain turns out to be completely isometrically isomorphic to the BPW completion of the algebra of the presentation, that is, $H^\infty_{\mathcal{R}}(G) = \tilde{\mathcal{A}} = \tilde{\mathcal{A}}_0$ completely isometrically. This theorem ties the theory of operator algebras of functions with the quantized function theory quite nicely. In particular, it allows us to extend the results to $H^\infty_{\mathcal{R}}(G)$ from the previous chapter.

Theorem 3.4.11. *Let G be an analytically presented domain with a separating presentation $\mathcal{R} = \{F_k : G \to M_{m_k,n_k} : k \in I\}$, let $\mathcal{A}$ be the algebra of the presentation and let $\widetilde{\mathcal{A}}$ be the BPW-completion of $\mathcal{A}$. Then*

(i) $\widetilde{\mathcal{A}} = H^\infty_{\mathcal{R}}(G)$, completely isometrically,

(ii) $H^\infty_{\mathcal{R}}(G)$ is a local weak-RFD dual operator algebra.*

Proof. Let $f \in M_n(\widetilde{\mathcal{A}})$, with $\|f\|_L < 1$. Then there exists a net of functions, $f_\lambda \in M_n(\mathcal{A})$, such that $\|f_\lambda\|_u < 1$ and $\lim_\lambda f_\lambda(z) = f(z)$ for every $z \in G$. Since $\|f_\lambda\|_\infty < 1$, by Montel's Theorem, there is a subsequence $\{f_n\}$ of this net that converges to f uniformly on compact sets. Hence, if $T \in \mathcal{Q}(G)$, then $\lim_n \|f(T) - f_n(T)\| = 0$ and so $\|f(T)\| \leq \sup_n \|f_n(T)\| \leq 1$. Thus, we have that $f \in M_n(H^\infty_{\mathcal{R}}(G))$, with $\|f\|_{\mathcal{R}} \leq 1$. This proves that $\widetilde{\mathcal{A}} \subseteq H^\infty_{\mathcal{R}}(G)$ and that $\|f\|_L \leq \|f\|_{\mathcal{R}}$.

Conversely, let $g \in M_n(H^\infty_{\mathcal{R}}(G))$ with $\|g\|_{\mathcal{R}} < 1$. Given any finite set $Y = \{y_1, ..., y_t\} \subseteq G$, let $\mathcal{A}/I_Y = \operatorname{span}\{E_1, ..., E_t\}$ be the corresponding t-idempotent algebra and let $\pi_Y : \mathcal{A} \to \mathcal{A}/I_Y$ denote the quotient map. Write $y_i = (y_{i,1}, ..., y_{i,N}), 1 \leq i \leq t$ and let $T_j = y_{1,j}E_1 + \cdots + y_{t,j}E_t, 1 \leq j \leq N$ so that $T = (T_1, ..., T_N)$ is a commuting N-tuple of operators with $\sigma(T) = Y$. For $k \in I$, we have that $\|F_k(T)\| = \|F_k(y_1) \otimes E_1 + \cdots + F_k(y_t) \otimes E_t\| = \|\pi_Y(F_k)\| \leq \|F_k\|_u = 1$. Thus, $T \in \mathcal{Q}(G)$, and so,

$$\|g(T)\| = \|g(y_1) \otimes E_1 + \cdots + g(y_t) \otimes E_t\| \leq \|g\|_{\mathcal{R}} < 1.$$

Since $\mathcal{A}$ separates points, we may pick $f \in M_n(\mathcal{A})$ such that $f = g$ on Y. Hence, $\pi_Y(f) = f(T) = g(T)$ and $\|\pi_Y(f)\| < 1$. Thus, we may pick $f_Y \in M_n(\mathcal{A})$, such that $\pi_Y(f_Y) = \pi_Y(f)$ and $\|f_Y\|_u < 1$. This net of functions, $\{f_Y\}$ converges to g pointwise and is bounded. Therefore, $g \in M_n(\widetilde{\mathcal{A}})$ and $\|g\|_L \leq 1$. This proves that $H^\infty_{\mathcal{R}}(G) \subseteq \widetilde{\mathcal{A}}$ and that $\|g\|_L \leq \|g\|_{\mathcal{R}}$.

Thus, $\widetilde{\mathcal{A}} = H^\infty_{\mathcal{R}}(G)$ and the two matrix norms are equal for matrices of all sizes. The rest of the conclusions follow from the results on BPW-completions. □

Remark 3.4.12. *The above result yields that for every $f \in H^\infty_{\mathcal{R}}(G)$, $\|f\|_{\mathcal{R}} = \sup\{\|\pi(f)\|\}$ where the supremum is taken over all finite dimensional weak*-continuous representations, $\pi : H^\infty_{\mathcal{R}}(G) \to M_n$ with n arbitrary. For many examples, we can show that $\|f\|_{\mathcal{R}} = \sup\{\|f(T)\| : T \in \mathcal{Q}_{00}(G)\}$ for any $f \in H^\infty_{\mathcal{R}}(G)$. Also, we can verify these hypotheses are met for most of the algebras given in the example section. In particular, for Examples 3.5.1, 3.5.2, 3.5.3, 3.5.4, 3.5.6, 3.5.7, and 3.5.8. It would be interesting to know if this can be done in general.*

Corollary 3.4.13. *Let G be an analytically presented domain with a separating presentation $\mathcal{R} = \{F_k : G \to M_{m_k,n_k} : k \in I\}$, let $\mathcal{A}$ be the algebra of the presentation and let $\widetilde{\mathcal{A}}$ be the BPW-completion of $\mathcal{A}$. If we assume that*

1. *$\mathcal{A}$ contains the coordinate functions, and*

2. *the Taylor spectrum of $T_\pi = (\pi(z_1), ..., \pi(z_n)) \subset G$, where π is a strict finite dimensional weak*-continuous completely contractive representation,*

then for every $f \in M_n(H^\infty_{\mathcal{R}}(G))$ we have that $\|f\|_{\mathcal{R}} = \sup\{\|f(T)\| : T \in \mathcal{Q}_{0,0}(G)\}$.

Proof. Let $\pi : H^\infty_{\mathcal{R}}(G) \to M_n$ be a weak*-continuous representation such that the Taylor spectrum of $T = (\pi(z_1), ..., \pi(z_N)) \subset G$. Then $\pi(f) = f(T)$ for every $f \in \mathcal{A}$ and for every $f \in H^\infty_{\mathcal{R}}(G)$ there exists a net of functions, $f_\lambda \in \mathcal{A}$ which converges to f in the BPW limit, since $\tilde{\mathcal{A}} = H^\infty_{\mathcal{R}}(G)$ completely isometrically. It follows from the Prop. 2.3.7 of Chapter 2 that $f_\lambda \to f$ in the weak*-topology. This further implies that $\pi(f_\lambda) = f_\lambda(T) \to \pi(f)$, and by the Taylor functional calculus $f_\lambda(T) \to f(T)$. Thus, we have that $\pi(f) = f(T)$ for every $f \in H^\infty_{\mathcal{R}}(G)$ and hence

$$\sup\{\|f(T)\| : T \in \mathcal{Q}_{0,0}(G)\} \geq \sup\{\|\pi(f)\|\}$$

where the supremum is taken over all finite dimensional weak*-continuous representations of $H^\infty_{\mathcal{R}}(G)$ such that the Taylor spectrum of $T = (\pi(z_1), ..., \pi(z_N)) \subset G$.

From the above theorem and the hypotheses, we get that for every $f \in H^\infty_{\mathcal{R}}(G)$, $\|f\|_{\mathcal{R}} = \sup\{\|\pi(f)\|\}$ where the supremum is taken over all finite dimensional weak*-continuous representations of $H^\infty_{\mathcal{R}}(G)$, $\pi : H^\infty_{\mathcal{R}}(G) \to M_n \ \forall\ n$. □

It is interesting to analyze the condition that the Taylor spectrum of $T_\pi = (\pi(z_1), ..., \pi(z_n)) \subseteq G$ where π is a finite dimensional weak*-continuous completely contractive homomorphism. In the following proposition, we prove that this condition is met if we assume that every complex homomorphism of the algebra of the presentation is given by a point evaluation map.

Proposition 3.4.14. *Let G be a analytically presented domain, let $\pi : H^\infty_{\mathcal{R}}(G) \to M_n$ be a finite dimensional weak*-continuous completely contractive homomorphism. If the algebra of the presentation $\mathcal{A}$ contains the coordinate functions and we assume that every weak*-continuous completely contractive complex homomorphism of $H^\infty_{\mathcal{R}}(G)$ is given by evaluation at some point in $\mathbb{C}^N$ where the functions in $\mathcal{R}$ are analytic and $T_i = \pi(z_i)$, then $T_\pi = (T_1, ..., T_N)$ is a commuting N-tuple of operators whose joint Taylor spectrum of T is contained in G.*

Proof. Note that $\pi(H^\infty_{\mathcal{R}}(G))$ is an abelian subalgebra of M_n, thus there exists an unitary $U \in M_n$ such that $\delta : H^\infty_R(G) \to T_n$ defined via $\delta(.) = U\pi(.)U^*$ is also a completely contractive and weak*-continuous homomorphism. By the unitary invariance property of a spectrum, we have that $\sigma(T_\pi) = \sigma((\delta(z_1), ..., \delta(z_N)))$. We denote $\delta(z_i)$ by S_i for every $i = 1, ..., N$. Since $S = (S_1, ..., S_N)$ is a commuting N-tuple of triangular matrices, the Taylor spectrum of S is equal to the joint point spectrum of S, that is, $\sigma(S) = \sigma(S_1) \times \cdots \times \sigma(S_N)$ where $\sigma(S_i)$ is the ordinary spectrum of a triangular matrix S_i for every $i = 1, ..., N$.

Fix $1 \leq k \leq N$, let e_k denote the column vector with 1 in the k-th position and 0 elsewhere. We define $\rho_k : H^\infty_{\mathcal{R}}(G) \to \mathbb{C}$ via $\rho_k(.) = \langle \delta(.)e_k, e_k \rangle$. It is easy to check that ρ_k is a weak*-continuous contractive homomorphism. Thus, by the hypthoses there exists a $\lambda_k \in \mathbb{C}^N$ such that $\rho_k(f) = f(\lambda_k)$ for every $f \in \mathcal{R}$. It follows from the Lemma 3.4.3 that $\|\rho(f_{ij})\| < 1$ for every $(f_{ij}) \in \mathcal{R}$. Thus, we have that $\sigma(T) = \{\lambda_1, ..., \lambda_N\} \subseteq G$. This completes the proof. $\square$

Remark 3.4.15. *We wish to justify one of our assumption in the previous result. In classical complex analysis of functions on several complex variables, the problem of finding a good characterization of complex domains G for which every weak*-continuous completely contractive complex homomorphism of $H^\infty(\Omega)$ is given by a point evaluation has been of* *interest since a long time. But as of this writing, no reasonable characterization of such domains is known. It is easy to see that it holds true for many examples such as* any *connected open set in* $\mathbb{C}$*, polydisk, and ball in* $\mathbb{C}^N$*. For an example where it fails, consider a shell in* $\mathbb{C}^N$*, in particular, take* $\Omega = \{z \in \mathbb{C}^2 : 1 < \|z\|_{\mathbb{C}^2} < 3\}$*. We refer the interested reader for more details to the book by Steven G. Krantz [50].*

Note that finding a characterization of an analytically presented domain G *for which every weak*-continuous completely contractive complex homomorphism of* $H^\infty_{\mathcal{R}}(G)$ *is given by a point evaluation map is even more difficult since* $H^\infty_{\mathcal{R}}(G) \subseteq H^\infty(G)$ *completely contractively.*

We now seek other characterizations of the functions in $H^\infty_{\mathcal{R}}(G)$. In particular, we wish to obtain analogues of Agler's factorization theorem and of the results in [10] and [17]. By Theorem 2.3.5, if we are given an analytically presented domain $G \subseteq \mathbb{C}^N$, with presentation $\mathcal{R} = \{F_k : G \to M_{m_k,n_k}, k \in I\}$, then there exist a Hilbert space $\mathcal{H}$ and a positive definite function, $K : G \times G \to B(\mathcal{H})$ such that $\widetilde{\mathcal{A}} = \mathcal{M}(K)$. We shall denote any kernel satisfying this property by $K_{\mathcal{R}}$.

Definition 3.4.16. *Let* $G \subseteq \mathbb{C}^N$ *be an analytically presented domain, with presentation* $\mathcal{R} = \{F_k : G^- \to M_{m_k,n_k}, k \in I\}$*. We shall call a function* $H : G \times G \to M_m$ *an* $\mathcal{R}$**-limit,** *provided that* H *is the pointwise limit of a net of functions* $H_\lambda : G \times G \to M_m$ *of the form given by Theorem 3.3.1(iii).*

Corollary 3.4.17. *Let* $G \subseteq \mathbb{C}^N$ *be an analytically presented domain, with a separating presentation* $\mathcal{R} = \{F_k : G^- \to M_{m_k,n_k}, k \in I\}$*. Then the following are equivalent:*

(i) $f \in M_m(H^\infty_{\mathcal{R}}(G))$ *and* $\|f\|_{\mathcal{R}} \le 1$,

(ii) $(I_m - f(z)f(w)^*) \otimes K_{\mathcal{R}}(z, w)$ *is positive definite,*

(iii) $I_m - f(z)f(w)^*$ *is an* $\mathcal{R}$*-limit.*

In the case when the presentation contains only finitely many functions we can say considerably more about $\mathcal{R}$-limits.

Proposition 3.4.18. *Let* G *be an analytically presented domain with a finite presentation* $\mathcal{R} = \{F_k = (f_{k,i,j}) : G \to M_{m_k,n_k}, 1 \leq k \leq K\}$. *For each compact subset* $S \subseteq G$, *there exists a constant* C, *depending only on* S, *such that given a factorization of the form,*

$$I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k=1}^{K} P_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)}P_k(w)^*,$$

then $\|P_k(z)\| \leq C$ *for all* $k \in I$ *and for all* $z \in S$.

Proof. By the continuity of the functions, there is a constant $\delta > 0$, such that $\|F_k(z)\| \leq 1 - \delta$, for all $k \in I$ and for all $z \in S$. Thus, we have that $I - F_k(z)F_k(z)^* \geq \delta I$, for all $k \in I$ and for all $z \in S$. Also, we have that

$$I_m \geq I_m - P(z)P(z)^* \geq P_k(z)(I - F_k(z)F_k(z)^*)^{(q_k)}P_k(z)^* \geq \delta P_k(z)P_k(z)^*.$$

This shows that $\|P_k(z)\| \leq 1/\delta$ for all $k \in I$ and for all $z \in S$. □

The proof of the following result is essentially contained in [17, Lemma 3.3].

Proposition 3.4.19. *Let* G *be a bounded domain in* $\mathbb{C}^N$ *and let* $F = (f_{i,j}) : G \to M_{m,n}$ *be analytic with* $\|F(z)\| < 1$ *for* $z \in G$. *If* $H : G \times G \to M_p$ *is analytic in the first variables, coanalytic in the second variables and there exists a net of matrix-valued functions* $P_\lambda \in M_{p,r_\lambda}(Hol(G))$ *which are uniformly bounded on compact subsets of* G, *such that* $H(z,w)$ *is the pointwise limit of* $H_\lambda(z,w) = P_\lambda(z)(I_m - F(z)F(w)^*)^{(q_\lambda)}P_\lambda(w)^*$ *where* $r_\lambda = q_\lambda m$, *then there exist a Hilbert space* $\mathcal{H}$ *and an analytic function,* $R : G \to B(\mathcal{H} \otimes \mathbb{C}^m, \mathbb{C}^p)$ *such that* $H(z,w) = R(z)[(I_m - F(z)F(w)^*) \otimes I_{\mathcal{H}}]R(w)^*$.

Proof. We identify $(I_m - F(z)F(w)^*)^{(q_\lambda)} = (I_m - F(z)F(w)^*) \otimes I_{\mathbb{C}^{q_\lambda}}$, and the $p \times mq_\lambda$ matrix-valued function P_λ as an analytic function, $P_\lambda : G \to B(\mathbb{C}^m \otimes \mathbb{C}^{q_\lambda}, \mathbb{C}^p)$. Writing $\mathbb{C}^m \otimes \mathbb{C}^{q_\lambda} = \mathbb{C}^{q_\lambda} \oplus \cdots \oplus C^{q_\lambda}$(m times) allows us to write $P_\lambda(z) = [P_1^\lambda(z), \ldots, P_m^\lambda(z)]$ where each $P_i^\lambda(z)$ is $p \times q_\lambda$. Also, if we let $f_1(z), ..., f_m(z)$ be the $(1, n)$ vectors that represent the rows of the matrix F, then we have that $F(z)F(w)^* = \sum_{i,j=1}^m f_i(z)f_j(w)^* E_{i,j}$.

Finally, we have that

$$H_\lambda(z, w) = \sum_{i=1}^m P_i^\lambda(z)P_i^\lambda(w)^* - \sum_{i,j=1}^m f_i(z)f_j(w)^* P_i^\lambda(z)P_j^\lambda(w).$$

Let $K_\lambda(z, w) = (P_i^\lambda(z)P_j^\lambda(w)^*)$, so that $K_\lambda : G \times G \to M_m(M_p) = B(\mathbb{C}^m \otimes \mathbb{C}^p)$, is a positive definite function that is analytic in z and co-analytic in w. By dropping to a subnet, if necessary, we may assume that K_λ converges uniformly on compact subsets of G to $K = (K_{i,j}) : G \times G \to M_m(M_p)$. Note that this implies that $P_i^\lambda(z)P_j^\lambda(w)^* \to K_{i,j}(z, w)$ for all i, j and that K is a positive definite function that is analytic in z and coanalytic in w.

The positive definite function K gives rise to a reproducing kernel Hilbert space $\mathcal{H}$ of analytic $\mathbb{C}^m \otimes \mathbb{C}^p$-valued functions on G. If we let $E(z) : \mathcal{H} \to \mathbb{C}^m \otimes \mathbb{C}^p$ be the evaluation functional, then $K(z, w) = E(z)E(w)^*$ and $E : G \to B(\mathcal{H}, \mathbb{C}^m \otimes \mathbb{C}^p)$ is analytic. Identifying $\mathbb{C}^m \otimes \mathbb{C}^p = \mathbb{C}^p \oplus \cdots \oplus \mathbb{C}^p$($m$ times), yields analytic functions, $E_i : G \to B(\mathcal{H}, \mathbb{C}^p), i = 1, ..., m$, such that $(K_{i,j}(z, w)) = K(z, w) = E(z)E(w)^* = (E_i(z)E_j(w)^*)$.

Define an analytic map $R : G \to B(\mathcal{H} \otimes \mathbb{C}^m, \mathbb{C}^p)$ by identifying $\mathcal{H} \otimes \mathbb{C}^m = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$($m$

times) and setting $R(z)(h_1 \oplus \cdots \oplus h_m) = E_1(z)h_1 + \cdots + E_m(z)h_m$. Thus, we have that

$$R(z)[(I_m - F(z)F(w)^*) \otimes I_{\mathcal{H}}]R(w)^* =$$

$$\sum_{i=1}^m E_i(z)E_i(w)^* - \sum_{i,j=1}^m f_i(z)f_j(w)^* E_i(z)E_j(w)^* =$$

$$\sum_{i=1}^m K_{i,i}(z,w) - \sum_{i,j=1}^m f_i(z)f_j(w)^* K_{i,j}(z,w) =$$

$$\lim_\lambda \sum_{i=1}^m P_i^\lambda(z)P_i^\lambda(w)^* - \sum_{i,j=1}^m f_i(z)f_j(w)^* P_i^\lambda(z)P_j^\lambda(w)^* = H(z,w),$$

and the proof is complete. □

Remark 3.4.20. *Conversely, any function that can be written in the form $H(z,w) = R(z)[(I_m - F(z)F(w)^*) \otimes I_{\mathcal{H}}]R(w)^*$ can be expressed as a limit of a net as above by considering the directed set of all finite dimensional subspaces of $\mathcal{H}$ and for each finite dimensional subspace setting $H_{\mathcal{F}}(z,w) = R_{\mathcal{F}}(z)[(I_m - F(z)F(w)^*) \otimes I_{\mathcal{F}}]R_{\mathcal{F}}(w)^*$, where $R_{\mathcal{F}}(z) = R(z)(P_{\mathcal{F}} \otimes I_m)$ with $P_{\mathcal{F}}$ the orthogonal projection onto $\mathcal{F}$.*

Definition 3.4.21. *We shall refer to a function $H : G \times G \to M_m(M_p)$ that can be expressed as $H(z,w) = R(z)[(I_m - F(z)F(w)^*) \otimes \mathcal{H}]R(w)^*$ for some Hilbert space $\mathcal{H}$ and some analytic function $R : G \to B(\mathcal{H} \otimes \mathbb{C}^m, \mathbb{C}^p)$, as an* **F-limit.**

Theorem 3.4.22. *Let G be an analytically presented domain with a finite separating presentation $\mathcal{R} = \{F_k = (f_{k,i,j}) : G \to M_{m_k,n_k}, 1 \le k \le K\}$, let $f = (f_{ij})$ be a $M_{m,n}$-valued function defined on G. Then the following are equivalent:*

(1) $f \in M_{mn}(H^\infty_{\mathcal{R}}(G))$ and $\|f\|_{\mathcal{R}} \le 1$,

(2) there exist an analytic operator-valued function $R_0 : G \to B(\mathcal{H}_0, \mathbb{C}^m)$ and F_k-limits, $H_k : G \times G \to M_m$, such that

$$I - f(z)f(w)^* = R_0(z)R_0(w)^* + \sum_{k=1}^K H_k(z,w) \;\forall\; z,w \in G,$$

(3) there exist F_k-limits, $H_k(z,w)$, such that

$$I - f(z)f(w)^* = \sum_{k=1}^{K} H_k(z,w) \ \forall z, w \in G.$$

Proof. Recall that $\tilde{\mathcal{A}} = H^\infty_{\mathcal{R}}(G)$. Let us first assume that $f \in M_{mn}(\tilde{A})$ and $\|f\|_{M_{mn}(\tilde{A})} < 1$. Then for each finite set Y, there exists $f_Y \in M_{mm}(\mathcal{A})$ such that f_Y converges to f pointwise and $\|f_Y\|_u \leq 1$. We may assume that $\|f_Y\|_u < 1$ by replacing f_Y by $\frac{f_Y}{1+1/|Y|}$, where $|Y|$ denotes the cardinality of the set Y.

Thus by Theorem 3.3.1 there exists a positive, invertible matrix $R^Y \in M_m$ and matrices $P_k^Y \in M_{m,r_{k_Y}}(\mathcal{A}), 0 \leq k \leq K$, such that

$$I_m - f_Y(z)f_Y(w)^* = R^Y + P_0^Y(z)P_0^Y(w)^* + \sum_{k=1}^{K} P_k^Y(z)(I - F_k(z)F_k(w)^*)^{(q_{k_Y})}P_k^Y(w)^*,$$

where $r_{k_Y} = q_{k_Y}m_k$ and $z,\ w \in G$. If we define a map $F_0 : G \to M_{m_0,n_0}$ as the zero map then the above factorization can be written as

$$I_m - f_Y(z)f_Y(w)^* = R^Y + \sum_{k=0}^{K} P_k^Y(z)(I - F_k(z)F_k(w)^*)^{(q_{k_Y})}P_k^Y(w)^*$$

where $r_{k_Y} = q_{k_Y}m_k$ and $z,\ w \in G$.

Note that the net R^Y is uniformly bounded above by 1, thus there exist $R \in M_m$ and a subnet R^{Y_s} which converges to R.

Finally, since the net f_Y converges to f pointwise we have that the net $\sum_{k=1}^{K} P_k^Y(z)(I - F_k(z)F_k(w)^*)^{(q_{k_Y})}P_k^Y(w)^*$ converges pointwise on G. Also note that for each k, $\{P_k^Y\}$ is a net of vector-valued holomorphic functions and is uniformly bounded on compact subsets of G by Proposition 3.4.18.

Thus by Proposition 3.4.19 there exist F_k-limits for each $0 \leq k \leq K$, that is, there exist $K+1$ Hilbert spaces $\mathcal{H}_k$ and $K+1$ analytic function, $R_k : G \to B(\mathcal{H}_k \otimes \mathbb{C}^M, \mathbb{C}^p)$ such that

$H_k(z,w) = R_k(z)[(I_m - F_k(z)F_k(w)^*) \otimes I_{\mathcal{H}_k}]R_k(w)^*$ and the corresponding subnet of the net $\sum_{k=0}^K P_k^Y(z)(I - F_k(z)F_k(w)^*)^{(q_{k_Y})} P_k^Y(w)^*$ converges to $\sum_{k=0}^K H_k(z,w)$ for all $z,\ w \in G$. This completes the proof that (1) implies (2).

To show the converse, assume that there exists an analytic operator-valued function $R_0 : G \to B(\mathcal{H}_0, \mathbb{C}^m)$ and K analytic functions, $R_k : G \to B(\mathcal{H}_k \otimes \mathbb{C}^M, \mathbb{C}^p)$ on some Hilbert spaces $\mathcal{H}_k$ such that $I - f(z)f(w)^* = R_0(z)R_0(w)^* + \sum_{k=1}^K R_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)} R_k(w)^*$ for every $z, w \in G$.

By using Theorem 2.3.5 there exists a vector-valued kernel K such that $M_n(\mathcal{M}(K)) = M_n(\tilde{A})$ completely isometrically for every n. It is easy to see that $((I - f(z)f(w)^*) \otimes K(z,w)) \geq 0$ for $z,\ w \in G$. This is equivalent to $f \in M_m(\mathcal{M}(K))$ and $\|M_f\| \leq 1$ which in turn is equivalent to (1). Thus, (1) and (2) are equivalent.

Clearly, (3) implies (2). The argument for why (2) implies (3) is contained in [17] and we recall it. If we fix any k_0, then since $\|F_{k_0}(z)\| < 1$ on G, we have that $|f_{k_0,1,1}(z)|^2 + \cdots + |f_{k_o,1,m}(z)|^2 < 1$ on G. From this it follows that $H(z,w) = (1 - f_{k_0,1,1}(z)\overline{f_{k_0,1,1}(w)} - \cdots - f_{k_0,1,m}(z)\overline{f_{k_0,1,m}(w)})$ is an F_{k_0}-limit and that $H^{-1}(z,w)$ is positive definite. Now we have that $R_0(z)R_0(w)^*H^{-1}(z,w)$ is positive definite and so we may write, $R_0(z)R_0(w)^*H^{-1}(z,w) = G_0(z)G_0(w)^*$ and we have that $R_0(z)R_0(w)^* = G_0(z)H(z,w)G(w)^*$. This shows that $R_0(z)R_0(w)^*$ is an F_k-limit and so it may be absorbed into the sum. □

3.5 Examples and Applications

In this section we present a few examples to illustrate the above definitions and results.

Example 3.5.1. *Let $G = \mathbb{D}^N$ be the polydisk which has a presentation given by the coordinate functions $F_i(z) = z_i,\ 1 \leq i \leq N$. Then the algebra of this presentation is* *the algebra of polynomials and an admissible representation is given by any choice of N commuting contractions, $(T_1, ..., T_N)$ on a Hilbert space. Given a matrix of polynomials, $\|(p_{i,j})\|_u = \sup \|(p_{i,j}(T_1, ..., T_N))\|$ where the supremum is taken over all N-tuples of commuting contractions. This is the norm considered by Agler in [2], which is sometimes called the* Schur-Agler *norm [51]. Our $\mathcal{Q}(\mathbb{D}^N) = \{T = (T_1, ..., T_N) : \sigma(T) \subseteq \mathbb{D}^N$ and $\|T_i\| \leq 1\}$. Note that if we replace such a T by $rT = (rT_1, ..., rT_N)$ then $\|rT_i\| < 1$, $rT \in \mathcal{Q}_{\mathcal{R}}(\mathbb{D}^N)$ and taking suprema over all $T \in \mathcal{Q}_{\mathcal{R}}(\mathbb{D}^N)$ will be the same as taking a suprema over this smaller set. Thus, the algebra $H^\infty_{\mathcal{R}}(\mathbb{D}^N)$ consists of those analytic functions f such that*

$$\|f\|_{\mathcal{R}} = \sup\{\|f(T_1, ..., T_N)\| : \|T_i\| < 1, i = 1, ..., N\} < +\infty.$$

Our result that this is a weak-RFD algebra shows that this supremum is also attained by considering commuting N-tuples of matrices satisfying $\|T_i\| < 1, i = 1, ..., N$. By Theorem 3.4.22 for $f \in M_{m,n}(H^\infty_{\mathcal{R}}(\mathbb{D}^N))$, we have that $\|f\|_{\mathcal{R}} \leq 1$ if and only if*

$$I_m - f(z)f(w)^* = \sum_{i=1}^{N}(1 - z_i\bar{w}_i)K_i(z, w),$$

for some analytic-coanalytic positive definite functions, $K_i : \mathbb{D}^N \times \mathbb{D}^N \to M_m$.

Example 3.5.2. *Let $G = \mathbb{B}_N$ denote the unit Euclidean ball in $\mathbb{C}^N$. If we let $F_1(z) = (z_1, ..., z_N) : \mathbb{B}_N \to M_{1,N}$, then this gives us a polynomial presentation. Again the algebra of the presentation is the polynomial algebra. An admissible representation corresponds to an N-tuple of commuting operators $(T_1, ..., T_n)$ such that $T_1T_1^* + \cdots + T_NT_N^* \leq I$, which is commonly called* a row contraction *and an admissible strict representation is given when $T_1T_1^* + \cdots + T_NT_N^* < I$, which is generally referred to as* a strict row contraction. *In this case one can again easily see that $\|\cdot\|_u = \|\cdot\|_{u_0}$ by using the same $r < 1$ argument as in the last example and that $f \in H^\infty_{\mathcal{R}}(\mathbb{B}_N)$ if and only if*

$$\|f\|_{\mathcal{R}} = \sup\{\|f(T)\| : T_1T_1^* + \cdots + T_NT_N^* < I\} < +\infty.$$

These are the norms on polynomials considered by Drury[39], Popescu [68], Arveson [15], and Davidson and Pitts [35].

Again our weak-RFD result shows that $\|f\|_{\mathcal{R}}$ is attained by taking the supremum over commuting N-tuples of matrices satisfying $T_1T_1^* + \cdots + T_NT_N^* < I$.*

By Theorem 3.4.22 we will have for $f \in M_{m,n}(H^\infty_{\mathcal{R}}(\mathbb{B}_N))$ that $\|f\|_{\mathcal{R}} \leq 1$ if and only if

$$I_m - f(z)f(w)^* = (1 - \langle z, w\rangle)K(z,w),$$

for some analytic-coanalytic positive definite functions, $K : \mathbb{B}_N \times \mathbb{B}_N \to M_m$.

Example 3.5.3. *Let $G = \mathbb{B}_N$ as above and let $F_1(z) = (z_1, ..., z_N)^t : \mathbb{B}_N \to M_{N,1}$. Again this is a polynomial presentation of G and the algebra of the presentation is the polynomials. An admissible representation corresponds to an N-tuple of commuting operators $(T_1, ..., T_N)$ such that $\|(T_1, ..., T_N)^t\| \leq 1$, i.e., such that $T_1^*T_1 + \cdots + T_N^*T_N \leq I$, which is generally referred to as a* column contraction. *This time the norm on $H^\infty_{\mathcal{R}}(\mathbb{B}_N)$ will be defined by taking suprema over all strict column contractions and we will have that $\|f\|_{\mathcal{R}} \leq 1$ if and only if*

$$I_m - f(z)f(w)^* = R_1(z)[(I_N - (z_i\bar{w}_j)) \otimes I_{\mathcal{H}}]R_1(w)^*$$

for some $R_1 : \mathbb{B}_N \to B(\mathbb{C}^m, \mathcal{H})$, analytic. Again, the weak-RFD result shows that $\|f\|_{\mathcal{R}}$ is attained by taking the supremum over matrices that form strict column contractions.*

Example 3.5.4. *Let $G = \mathbb{B}_N$ as above, let $F_1(z) = (z_1, ..., z_N) : \mathbb{B}_N \to M_{1,N}$ and $F_2(z) = (z_1, ..., z_N)^t : \mathbb{B}_N \to M_{N,1}$. Again this is a polynomial presentation of G and the algebra of the presentation is the polynomials. An admissible representation corresponds to an N-tuple of commuting operators $(T_1, ..., T_N)$ such that $T_1T_1^* + \cdots + T_NT_N^* \leq I$ and $T_1^*T_1 +$*

$\cdots + T_N^* T_N \leq I$, that is, which is both a row and column contraction. This time the norm on $H^\infty_{\mathcal{R}}(\mathbb{B}_N)$ is defined as the supremum over all commuting N-tuples that are both strict row and column contractions and again this is attained by restricting to commuting N-tuples of matrices that are strict row and column contractions. We will have that $f \in M_{m,n}(H^\infty_{\mathcal{R}}(\mathbb{B}_N))$ with $\|f\|_{\mathcal{R}} \leq 1$ if and only if

$$I_m - f(z)f(w)^* = (1 - \langle z, w\rangle)K_1(z,w) + R_1(z)[(I_N - (z_i\bar{w}_j)) \otimes I_{\mathcal{H}}]R_1(w)^*,$$

where K_1 and R_1 are as before.

The last three examples illustrate that it is possible to have multiple polynomial representations of G, all with the same algebra, but which give rise to (possibly) different operator algebra norms on $\mathcal{A}$. Thus, the operator algebra norm depends not just on G, but also on the particular presentation of G that one has chosen. We have suppressed this dependence on $\mathcal{R}$ to keep our notation simplified.

Example 3.5.5. *Let $G = \mathbb{D}$ the open unit disk in the complex plane and let $F_1(z) = z^2, F_2(z) = z^3$. It is easy to check that the algebra $\mathcal{A}$ of this presentation is generated by the component functions and the constant function so that $\mathcal{A}$ is the span of the monomials, $\{1, z^n : n \geq 2\}$. Also, $\mathcal{A}$ separates the points of G. In this case an (strict)admissible representation, $\pi : \mathcal{A} \to B(\mathcal{H})$, is given by any choice of a pair of commuting (strict)contractions, $A = \pi(z^2), B = \pi(z^3)$, satisfying $A^3 = B^2$. Again, it is easy to see that $\|.\|_u = \|.\|_{u_0}$. On the other hand*

$$\mathcal{Q}(\mathbb{D}) = \{T : \sigma(T) \subseteq \mathbb{D} \text{ and } \|T^2\| \leq 1, \|T^3\| \leq 1\}$$

and it can be seen that $H^\infty_{\mathcal{R}}(\mathbb{D})$ is defined by

$$\|f\|_{\mathcal{R}} = \sup\{\|f(T)\| : \|T^2\| < 1, \|T^3\| < 1\} < +\infty.$$

In this case we have that $f \in M_{m,n}(H^\infty_{\mathcal{R}}(\mathbb{D}))$ and $\|f\|_{\mathcal{R}} \leq 1$ if and only if

$$I_m - f(z)f(w)^* = (1 - z^2\overline{w}^2)K_1(z,w) + (1 - z^3\overline{w}^3)K_2(z,w).$$

However, our weak-RFD result only guarantees that $\|f\|_{\mathcal{R}}$ is attained by taking the supremum all finite dimensional representations π such that $\pi(z^2) = A$ and $\pi(z^3) = B$ are commuting strict contractions satisfying $A^3 = B^2$. However, given such a pair there is, in general, no single matrix T such that $T^2 = A$ and $T^3 = B$. So our results do not guarantee, that $\|f\|_{\mathcal{R}}$ is attained by taking the supremum over all matrices T satisfying $\|T^2\| < 1$ and $\|T^3\| < 1$.*

Example 3.5.6. *Let $\mathbb{L} = \{z \in \mathbb{C} : |z - a| < 1, |z - b| < 1\}$, where $|a - b| < 1$, then the functions $f_1(z) = z - a, f_2(z) = z - b$ give a polynomial presentation of this "lens". The algebra of this presentation is again the algebra of polynomials. An admissible representation of this algebra is defined by choosing any operator satisfying $\|T - aI\| \leq 1$ and $\|T - bI\| \leq 1$, with strict inequalities for the admissible strict representations. In this case we easily see that $\|\cdot\|_u = \|\cdot\|_{u_0}$, since given any operator T satisfying $\|T - aI\| \leq 1$ and $\|T - bI\| \leq 1$, and $r < 1$, $S_r = rT + (1 - r)(a + b)$ corresponds to the admissible strict representations and for any matrix of polynomials $\|(p_{i,j}(T))\| = \lim_{r \to 1} \|(p_{ij}(S_r))\|$. This algebra with this norm was studied in [21]. Their work shows that this norm is completely boundedly equivalent to the usual supremum norm and consequently, $H^\infty_{\mathcal{R}}(\mathbb{L}) = H^\infty(\mathbb{L})$, as sets, but the norms are different.*

Our results imply that $f \in M_{m,n}(H^\infty_{\mathcal{R}}(\mathbb{L}))$ and $\|f\|_{\mathcal{R}} \leq 1$ if and only if

$$I_m - f(z)f(w)^* = (1 - (z - a)(\overline{w - b}))K_1(z,w) + (1 - (z - b)(\overline{w - b}))K_2(z,w).$$

Since the coordinate function z belongs to the algebra $\mathcal{A}$, our weak-RFD results again show that $\|f\|_{\mathcal{R}}$ is attained by choosing matrices satisfying $\|T - aI\| < 1, \|T - bI\| < 1$.*

Example 3.5.7. *Let $G = \{(z_{i,j}) \in M_{M,N} : \|(z_{i,j})\| < 1\}$ and let $F : G \to M_{M,N}$ be the identity map $F(z) = (z_{i,j})$. Then this is a polynomial presentation of G and the algebra of the presentation is the algebra of polynomials in the MN variables $\{z_{i,j}\}$. An admissible representation of this algebra is given by any choice of MN commuting operators $\{T_{i,j}\}$ on a Hilbert space $\mathcal{H}$, such that $\|(T_{i,j})\| \le 1$ in $M_{M,N}(B(\mathcal{H}))$ and as above, one can show that $\|\cdot\|_{\mathcal{R}}$ is achieved by taking suprema over all commuting MN-tuples of matrices for which $\|(T_{i,j})\| < 1$. We have that $f \in M_{m,n}(H^\infty_{\mathcal{R}}(G)$ and $\|f\|_{\mathcal{R}} \le 1$ if and only if*

$$I_m - f(z)f(w)^* = R_1(z)[(I_M - (z_{i,j})(w_{i,j})^*) \otimes I_{\mathcal{H}}]R_1(w)^*,$$

for some appropriately chosen R_1.

All of the above examples are also covered by the theory of [10] and [17], except that their definition of the norm is slightly different and the fact that the suprema are attained over matrices rather than operators, i.e., the weak*-RFD consequences, seem to be new. We address the difference between their definition of the norm and ours in a later remark. We now turn to some examples that are not covered by these other theories.

Example 3.5.8. *Let $0 < r < 1$ be fixed and let $\mathbb{A}_r = \{z \in \mathbb{C} : r < |z| < 1\}$ be an annulus. Then this has a rational presentation given by $F_1(z) = z$ and $F_2(z) = rz^{-1}$, and the algebra of this presentation is just the Laurent polynomials. Admissible representations of this algebra are given by selecting any invertible operator T satisfying $\|T\| \le 1$ and $\|T^{-1}\| \le r^{-1}$. Admissible strict representations are given by invertible operators satisfying $\|T\| < 1$ and $\|T^{-1}\| < r^{-1}$. The algebra that we denote $H^\infty_{\mathcal{R}}(\mathbb{A}_r)$ is also introduced in [5] where it is called the* Douglas-Paulsen algebra.

It is no longer immediate that $\|\cdot\|_u = \|\cdot\|_{u_0}$. *However, this algebra with these norms are studied more carefully in the last chapter and among other results the equality of these norms was shown. Consequently,* $\|f\|_{\mathcal{R}}$ *is attained by taking the supremum over matrices* T *satisfying* $\|T\| < 1$ *and* $\|T^{-1}\| < r^{-1}$.

The formula for the norm is given by $\|f\|_{\mathcal{R}} \leq 1$ *if and only if*

$$I_m - f(z)f(w)^* = (1 - z\bar{w})K_1(z,w) + (1 - r^2 z^{-1}\overline{w^{-1}})K_2(z,w).$$

The scalar version of this formula is also shown in [5].

Douglas and Paulsen showed in [35] that $\|\cdot\|_u$ *is completely boundedly equivalent to the usual supremum norm, but that the two norms are not equal. In fact, they exhibit an explicit function for which the norms are different. Since the norms are equivalent, it follows that* $H^\infty_{\mathcal{R}}(\mathbb{A}_r) = H^\infty(\mathbb{A}_r)$ *as sets. Badea, Beckermann and Crouzeix [16] show that not only are the norms equivalent, but that there is a universal constant* C, *independent of* r, *such that* $\|f\|_\infty \leq \|f\|_{\mathcal{R}} \leq C\|f\|_\infty$.

Example 3.5.9. *Let* G *be a simply connected domain in* $\mathbb{C}$ *and* $\phi : G \to \mathbb{D}$ *be a biholomorphic map. Then* $G = \{z \in \mathbb{C} : |\phi(z)| < 1\}$ *and* $\mathcal{Q}(G) = \{T : \sigma(T) \subseteq G$ *and* $\|\phi(T)\| \leq 1\}$ *where* $\mathcal{R} = \{\phi\}$. *In this case the algebra* $\mathcal{A}$ *of the presentation is just the algebra of all polynomials in* ϕ, *regarded as a subalgebra of the algebra of analytic functions on* G. *Thus, an admissible representation of this algebra is defined by choosing an operator* $B \in B(\mathcal{H})$ *that satisfies* $\|B\| \leq 1$ *and defining* $\pi : \mathcal{A} \to B(\mathcal{H})$ *via* $\pi(p(\phi)) = p(B)$, *where* p *is a polynomial. A strict admissible representation is defined similarly by first choosing a strict contraction. In this case, it is immediate that* $\|.\|_u = \|.\|_{u_0}$ *and that* $f \in H^\infty_{\mathcal{R}}(G)$ *if and only if* $f \in Hol(G)$ *and*

$$\|f\|_{\mathcal{R}} = \sup\{\|f(T)\| : T \in \mathcal{Q}_{\mathcal{R}}(G)\} < +\infty.$$

Our results imply that $\|f\|_{\mathcal{R}} \leq 1$ *if and only if*

$$1 - f(z)f(w)^* = (1 - \phi(z)\phi(w)^*)K(z,w).$$

In particular, if we take $\phi(z) = \frac{1-z}{1+z}$ *then it maps the half plane* $\mathbb{H} = \{z : Re(z) > 0\}$ *onto the unit disk. For this particular* ϕ*, we have that* $\mathcal{Q}_{\mathcal{R}} = \{T : \sigma(T) \subseteq \mathbb{H}$ *and* $Re(T) \geq 0\}$.

Example 3.5.10. *Similarly, if we let* $\mathbb{H} = \{z \in \mathbb{C}^N : |\phi_i(z)| < 1, i = 1, ..., N\}$ *where each* $\phi_i(z) = \frac{1-z_i}{1+z_i}$*, then* G *is an intersection of half planes and* $\mathcal{Q}_{\mathcal{R}}$ *consists of all commuting* N*-tuples of operators,* $(T_1, ..., T_N)$ *such that* $\sigma(T_i) \subseteq \mathbb{H}$ *and* $Re(T_i) \geq 0$ *for all* i*. Applying our results, we obtain a factorization result for half planes. These algebras have been studied by D. Kalyuzhnyi-Verbovetzkii in [48]. In this case, the algebra of the presentation is the algebra of polynomials in* N*-variables:* $\phi_1, \cdots, \phi_N$*. In other words,* $\mathcal{A}$ *is the linear span of the elements of the type* $\phi_{i_1}^{l_1}\phi_{i_2}^{l_2}...\phi_{i_N}^{l_N}$*. Thus, an admissible representation of this algebra is defined by choosing a tuple of operators* $(S_1, \ldots, S_N)$ *such that each* S_i *is an operator on some Hilbert space,* $\mathcal{H}$ *and* $\|S_i\| \leq 1$*. Given* $r < 1$*, we may define* $\pi_r : \mathcal{A} \to B(\mathcal{H})$ *via*

$$\pi_r(\phi_{i_1}^{l_1}\phi_{i_2}^{l_2}...\phi_{i_N}^{l_N}) = r^{l_1+l_2+...+l_n} {S_{i_1}}^{l_1} {S_{i_2}}^{l_2} ... {S_{i_N}}^{l_N}.$$

Note that π_r *is a well-defined map for each* $r < 1$*. Indeed, it follows from the fact that every finite subset of* $\{\phi_1^{l_1}\phi_2^{l_2}...\phi_N^{l_N} : l_j \in \mathbb{N} \cup \{0\},\ 1 \leq j \leq N\}$ *is a linearly independent set which in turn follows from the easy fact that the set* $\{w_1^{l_1}w_2^{l_2}...w_N^{l_N} : l_j \in \mathbb{N} \cup \{0\}\ \forall\ 1 \leq j \leq N\}$ *is a linearly independent set where each* $w_i \in \mathbb{D}$*. By extending this map linearly on* $\mathcal{A}$*, we get a strict admissible representation for each* $r < 1$ *such that* $\lim_{r \nearrow 1} \pi_r(f) = \pi(f)$*. Now, it is straightforward to conclude that* $\|.\|_u = \|.\|_{u_0}$ *for matrices of all sizes.*

Example 3.5.11. *Let* $G \subseteq \mathbb{C}$ *be an open convex set and represent it as an intersection of half planes* $\mathbb{H}_\theta$*. Each half plane can be expressed as* $\{z : |F_\theta(z)| < 1\}$ *for some family of linear fractional maps. If we let* $\mathcal{R} = \{F_\theta\}$*, then* $\mathcal{Q}_{\mathcal{R}}(G) = \{T : \sigma(T) \subseteq G$ *and* $\|F_\theta(T)\| \leq$

$1\ \forall\theta\}$. Moreover, each inequality $\|F_\theta(T)\| \le 1$ can be re-written as an operator inequality for the real part of some translate and rotation of T. For example, when $G = \mathbb{D}$, we may take $F_\theta(z) = \frac{z}{z-2e^{i\theta}}$, for $0 \le \theta < 2\pi$. In this case, one checks that $\|F_\theta(T)\| \le 1$ if and only if $Re(e^{i\theta}T) \le I$. Thus, it follows that

$$\mathcal{Q}(\mathbb{D}) = \{T : \sigma(T) \subseteq \mathbb{D} \text{ and } w(T) \le 1\},$$

where $w(T)$ denotes the numerical radius of T. Thus, $H^\infty_{\mathcal{R}}(\mathbb{D})$ becomes the "universal" operator algebra that one obtains by substituting an operator of numerical radius less than one for the variable z and we have a quite different quantization of the unit disk. Our results give a formula for this norm, but only in terms of $\mathcal{R}$-limits, so further work would need to be done to make it explicit.

Example 3.5.12. *There is a second way that one can quantize many convex sets. Let $G = \{z : |z - a_k| < r_k, k \in I\} \subseteq \mathbb{C}$ be an open, bounded convex set that can be expressed as an intersection of a possibly infinite set of open disks. For example, the open unit square can not be expressed as such an intersection, but any convex set with a smooth boundary with uniformly bounded curvature can be expressed in such a fashion. Then G has a rational presentation given by $F_k(z) = r_k^{-1}(z - a_k), k \in I$ the algebra of the presentation is just the polynomial algebra and an admissible representation is given by selecting any operator $T \in B(\mathcal{H})$ satisfying, $\|T - a_k I\| \le r_k, k \in I$ and a strict admissible representation is defined similarly using a strict contraction. If we take $r < 1$, then $T_r = rT$ is a strict contraction such that $\|rT - a_k\| \le rr_k < r_k$ which further implies that for each $r < 1$ we get a strict admissible representation $\pi_r : \mathcal{A} \to B(\mathcal{H})$ which can be defined via the map $\pi_r(f) = f(r\pi(z))$. From this it follows that $\|.\|_u = \|.\|_{u_0}$. Note, we again get a factorization result, but only in terms of $\mathcal{R}$-limits.*

The above definitions allow one to consider many other examples. For example, one could fix $0 < r < 1$ and let $G = \{z \in \mathbb{B}_N : r < |z_1|\}$, with rational presentation $f_1(z) = (z_1, ..., z_N) \in M_{1,N}$, and $f_2(z) = rz_1^{-1}$. An admissible representation would then correspond to a commuting row contraction with T_1 invertible and $\|T_1^{-1}\| \leq r^{-1}$.

We now compare and contrast some of our hypotheses with those of [10] and [17].

Remark 3.5.13. *Let $G = \{z \in \mathbb{C}^N : \|F_k(z)\| < 1, k = 1, \cdots, K\}$ where the F_k's are matrix-valued polynomials defined on G. Then for $f \in Hol(G)$, [10] and [17] really study a norm given by $\|f\|_s = sup\{\|f(T)\|\}$ where the supremum is taken over all commuting N-tuples of operators T with $\|F_k(T)\| < 1$, for $1 \leq k \leq K$. We wish to contrast this norm with our $\|f\|_{\mathcal{R}}$. In [10] it is shown that the hypotheses $\|F_k(T)\| < 1, k = 1, ..., K$ implies that $\sigma(T) \subseteq G$. Thus, we have that $\|f\|_s \leq \|f\|_{\mathcal{R}}$. In fact, we have that $\|f\|_s = \|f\|_{\mathcal{R}}$. This can be seen by the fact that they obtain identical factorization theorems to ours. This can also be seen directly in some cases where the algebra $\mathcal{A}$ contains the polynomials and when it can be seen that $\|\cdot\|_{\mathcal{R}}$ is attained by taking the supremum over matrices(see Remark 3.4.12). Indeed, if $\|f\|_{\mathcal{R}}$ is attained as the supremum over commuting N-tuples of finite matrices $T = (T_1, ..., T_N)$ satisfying $\sigma(T) \subseteq G$ and $\|F_k(T)\| \leq 1$ then such an N-tuple of commuting matrices, can be conjugated by a unitary to be simultaneously put in upper triangular form. Now it is easily argued that the strictly upper triangular entries can be shrunk slightly so that one obtains new N-tuples $T_\epsilon = (T_{1,\epsilon}, ..., T_{N,\epsilon})$ satisfying, $\|F_k(T_\epsilon)\| < 1, k = 1, ..., K$ and $\|T_i - T_{i,\epsilon}\| < \epsilon$. But we do not have a simple direct argument that works in all cases.*

Remark 3.5.14. *We do not know how generally it is the case that $\|\cdot\|_u$ is a local norm. That is, we do not know if $\|f\|_u = \|f\|_{\mathcal{R}}$ for $f \in M_n(\mathcal{A})$. In particular, we do not know if this is the case for Example 3.5.5. In this case, the algebra of the of the presentation is $\mathcal{A} = span\{z^n : n \geq 0, n \neq 1\}$. If we write a polynomial $p \in \mathcal{A}$ in terms of its even and odd* *decomposition,* $p = p_e + p_o$*, then* $p_e(z) = q(z^2)$ *and* $p_o = z^3 r(z^2)$ *for some polynomials* q, r*. In this case it is easily seen that*

$$\|p\|_u = \sup\{\|q(A) + Br(A)\| : \|A\| \leq 1, \|B\| \leq 1, AB = BA, A^3 = B^2\},$$

while

$$\|p\|_L = \|p\|_{\mathcal{R}} = \sup\{\|p(T)\| : \|T^2\| \leq 1, \|T^3\| \leq 1\}.$$

Chapter 4

Fejér Kernels

4.1 Introduction

The Fejér kernel is a trigonometric polynomial that first arose in Fourier analysis. It is named after the brilliant Hungarian mathematicians Lipót Fejér. In 1900, when Fejér was only 20 years old, he made a fundamental discovery by expressing the effect of Cesàro summation on the Fourier series. The Cesàro means of a special Fourier series which came up in this discovery, turned out to be extremely useful and gives rise to the Fejér kernel.

If f is a complex-valued Lebesgue-integrable function on the unit circle $\mathbb{T}$, then the Fourier series for f is the formal power series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, where the complex numbers c_n's are called Fourier coefficients of f and are given by

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}dx$$

The n-th partial sum of the series is given by

$$s_n(x; f) = \sum_{k=-n}^{n} c_n e^{inx}$$

and the Cesàro means are the arithmetic means of these partial sums. The n-th Cesàro mean of the function f is given by

$$\sigma_n(x; f) = \frac{1}{n}(s_0(x; f) + s_1(x; f) + \cdots + s_{n-1}(x; f)).$$

The obvious desire is to be able to recapture f from its Fourier series. One might hope that the partial sums of the Fourier series of a continuous functions f converge uniformly to a continuous function. To everybody's surprise, in 1876 Du Bois-Reymond constructed a continuous function whose Fourier series diverges somewhere. Then in 1900, Fejér suggested his method of arithmetic mean summation and saw that although partial sums of a Fourier series could fail to converge, their averages might behave rather better. He proved the remarkable result that if f is a 2π periodic continuous function then the Cesàro means of its Fourier series converges to f uniformly. This is often referred to as Fejér's Theorem. In the proof of this theorem, Fejér made use of an integral representation of the Cesàro means of the Fourier series of f. This involved a sequence of functions F_n's in a way so that

$$\sigma_n(x; f) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)F_n(t)dt = F_n * f,$$

where $*$ is the convolution product and $F_n(t) = \sum_{k=-n}^{n} \frac{1}{2\pi}(1 - \frac{|k|}{n+1})e^{ikt}$ is called the *Fejér kernel* which happens to be the n-th Cesàro mean of the Fourier series $\sum_{k=-\infty}^{\infty} e^{ikt}$. This kernel function has remarkable properties which makes it an extremely useful in the study of Fourier series. In fact, F_n turns out to be an approximate identity for the space of integrable functions on the unit circle with respect to the convolution product. We record some of the properties of the Fejér kernel here which are relevant to us:

1. $F_n \geq 0$,

2. $\frac{1}{2\pi}\int_{\mathbb{T}} F_n(t)dt = 1$,

3. $F_n(t) = \frac{1}{n+1}\sum_{k,l=0}^{n} e^{i(k-l)t}$.

A more detailed description of Fejér kernel and its properties can be found in most of the books on classical harmonic analysis. We refer the reader to [45].

We now turn towards the usefulness of the kernel function in our context. In our joint paper [51] with Lata and Paulsen, we gave a proof of Agler's factorization for the polydisk without appealing to the theory of operator algebras of functions. Instead, we used the idea of approximation by using the Fejér kernel, an idea that was suggested to us by S. McCullough. In this chapter, we wish to outline that proof and exhibit the importance of Fejér kernel methods by giving an elementary proof of the GNFT for certain other domains as well.

4.2 Application of Fejér kernels

Let $G \subseteq \mathbb{C}^N$ be an analytically presented domain and let $\mathcal{A}$ be the algebra of the presentation. In this section we present an application of the existence of the approximating sequence of functions in the algebra of the presentation for certain analytically presented domains by using Fejér kernel theory.

We have divided this section into two subsections to be able to treat two different sets of examples separately. In the first section we treat a class of examples which share similar characteristics, that is, the unit balls of some norm in $\mathbb{C}^N$, and in the second section we present the similar study for the domain which possess some distinctively different properties, that is, the annulus.

4.2.1 Balls in $\mathbb{C}^N$

Let $G \subseteq \mathbb{C}^N$ be the unit ball of some norm and we assume that G has some analytic presentation for which the algebra of the presentation is the algebra of polynomials.

Fix an $n \in \mathbb{N}$ and consider the Fejér kernel,

$$F_n(\theta) = \tfrac{1}{n+1} \textstyle\sum_{k,l=0}^{n} e^{i(k-l)\theta} \text{ for } \theta \in [0, 2\pi].$$

Recall that $F_n(\theta) \geq 0$ and $\frac{1}{2\pi} \int_0^{2\pi} F_n(\theta) = 1$.

Given an analytic function $f : G \to \mathbb{C}$, we define

$$\phi_n(f)(z_1, ..., z_N) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} z_1, ..., e^{i\theta} z_N) F_n(\theta) d\theta,$$

so that $\phi_n(f)$ is a polynomial of total degree n.

Using the power series expansion of f about 0, it is easy to check that the sequence of polynomials $\phi_n(f)$ converges pointwise to f on G and that at each point, $|\phi_n(f)(z)| \leq |f(z)|$. Thus, when f is bounded, we have that $\|\phi_n(f)\|_\infty \leq \|f\|_\infty$.

Recall that there exist three norms on the algebra of any presentation: $\|\cdot\|_{u_0}$, $\|\cdot\|_u$, $\|\cdot\|_{\mathcal{R}}$. Let $f \in \mathcal{A}$, then

$$\|f\|_u = \sup\{\|\pi(f)\|\}$$

where the supremum is taken over all admissible representations $\pi : \mathcal{A} \to B(\mathcal{H})$ and all Hilbert spaces $\mathcal{H}$,

$$\|f\|_{u_0} = \sup\{\|\pi(f)\|\}$$

where the supremum is taken over all strict admissible representations $\pi : \mathcal{A} \to B(\mathcal{H})$ and all Hilbert spaces $\mathcal{H}$, and

$$\|f\|_{\mathcal{R}} = \sup\{\|f(T)\|\}$$

where the supremum is taken over all commuting tuple of operators $T = (T_1, \cdots, T_N)$ such that $\|F_k(T)\| \leq 1$ and $\sigma(T) \subseteq G$.

Proposition 4.2.1. *Let $G \subseteq \mathbb{C}^N$ be the open unit ball for some norm and assume that G has an analytic presentation given by a set of non-constant functions, $F_k : G \to M_{m_k,n_k}$, $k \in I$, where I is the indexing set. Then $\|f\|_{u_0} \leq \|f\|_{\mathcal{R}} \leq \|f\|_u$ for every $f \in \mathcal{A}$.*

Proof. Note that the inequality $\|f\|_{\mathcal{R}} \leq \|f\|_u$ is immediate. Since the algebra of the presentation is the algebra of the polynomial and contains all the coordinates functions, we have by the Proposition 3.4.14 that $\|f\|_{u_0} \leq \|f\|_{\mathcal{R}}$. This completes the proof of this result. □

In the next proposition, we prove that all three norms on $\mathcal{A}$ are equal under an additional reasonable hypothesis. We know that every unit ball G in $\mathbb{C}^N$ is circular, i.e., $e^{i\theta}G \subseteq G$ for every $\theta \in [0, 2\pi]$. It is quite natural to expect the same for $\mathcal{Q}(G)$. One way to impose this condition is to require F_θ defined by $F_\theta(.) := F(e^{i\theta}.)$ belong to $\mathcal{R}$ for every $F \in \mathcal{R}$ or $F_\theta \in H^\infty_{\mathcal{R}}(G)$ and $\|F_\theta\| \leq 1$. Note that the former entirely depends on the choice of the presentation. On the other hand, the latter is equivalent to the condition that $\mathcal{Q}(G)$ is circular. Unfortunately, we haven't been able to prove this condition, so we make this an assumption to prove the following result.

Proposition 4.2.2. *Let $G \subseteq \mathbb{C}^N$ be the open unit ball for some norm and assume that G has an analytic presentation given by a set of non-constant functions, $F_k : G \to M_{m_k,n_k}$, $k \in I$, where I is the indexing set and that $e^{i\theta}\mathcal{Q}(G) \subseteq \mathcal{Q}(G)$ for every $\theta \in [0, 2\pi]$. Then, $\mathcal{A} = \mathcal{A}_0$ completely isometrically and thus the inclusion of $\mathcal{A}$ into $H^\infty_{\mathcal{R}}(G)$ is completely isometric.*

Proof. Since the defining functions are all non-constant, we have by the maximum modulus theorem and the hypothesis that for each $0 < r < 1$, $\|F_k(rT)\| \leq \delta_r < 1, \forall\ k$.

From this it follows that, if π is an admissible representation such that $\pi(z_i) = T_i$ then $\pi_r(z_i) = rT_i$ defines a strict admissible representation. Hence, for each $(f_{i,j}) \in M_n(\mathcal{A})$, $\|(f_{i,j}(T))\|_u \leq \sup_{0<r<1}\{\|(f_{i,j}(rT))\|\} \leq \|(f_{i,j})\|_{u_0}$. Thus, $\|(f_{i,j})\|_u \leq \|(f_{i,j})\|_{u_0}$, and equality of all three operator algebra norms on $\mathcal{A}$ follows. □

Remark 4.2.3. *If we assume that G is a complete circular domain, that is, for every $z \in G$ and for every λ such that $\|\lambda\| \leq 1$ we have that $\lambda z \in G$ so that its quantized version is also a complete circular domain. Then by mimicking the above proof, we can prove that $\mathcal{A} = \mathcal{A}_0$ completely isometrically, where $\mathcal{A}$ is an algebra of the presentation of G with a presentation given by non-constant functions.*

Theorem 4.2.4. *Let $G \subseteq \mathbb{C}^N$ be the open unit ball for some norm and assume that G has an analytic presentation given by a set of non-constant functions, $F_k : G \to M_{m_k,n_k}$ and that $e^{i\theta}\mathcal{Q}(G) \subseteq \mathcal{Q}(G)$ for every $\theta \in [0, 2\pi]$. Let $F \in \mathcal{M}_{m,n}(H^\infty_{\mathcal{R}}(G))$ with $F(z) = \sum_I A_I z^I$ for $z \in G$. Then the sequence of matrices of polynomials $\phi_n(F)(z) = \sum_{|I|\leq n}(1 - \frac{|I|}{n+1})A_I z^I$ converges locally uniformly to F and $\|\phi_n(F)\|_{\mathcal{R}} \leq \|F\|_{\mathcal{R}}$ for each n. Conversely, if there is a sequence of matrices of polynomials $\phi_n(F)$ converging to F pointwise on G with $\|\phi_n(F)\|_{\mathcal{R}} \leq 1$ for each n, then $\|F\|_{\mathcal{R}} \leq 1$.*

Proof. Fix an $n \in \mathbb{N}$ and consider the Fejér kernel,

$$F_n(\theta) = \tfrac{1}{n+1}\textstyle\sum_{k,l=0}^{n} e^{i(k-l)\theta} \text{ for } \theta \in [0, 2\pi].$$

Note that for each fixed $z \in G$ the function $\theta \mapsto F(ze^{i\theta}) = F(z_1 e^{i\theta}, \ldots, z_N e^{i\theta})$ is continuous. We define $\phi_n(F)(z) = \frac{1}{2\pi}\int_0^{2\pi} F(ze^{i\theta})F_n(e^{i\theta})d\theta$ for every $z \in G$, where the integration is in the Riemann sense. A direct calculation shows that $\phi_n(F)(z) = \sum_{|I|\leq n}(1 - \frac{|I|}{n+1})A_I z^I$, where $|I| = i_1 + \cdots + i_N$.

Next check that for a fixed commuting N-tuple of operators, $T = (T_1, \ldots, T_N) \in \mathcal{Q}(G)$, on a Hilbert space $\mathcal{H}$, the map $\theta \mapsto F(T_1 e^{i\theta}, \ldots, T_N e^{i\theta})$ is continuous from the interval into $B(\mathcal{H})$ equipped with the norm topology. This follows from the fact that $\sigma(e^{i\theta}T) \subset G$, $F(Te^{i\theta})$ is a norm limit of partial sums of its power series. It now follows that

$$\phi_n(F)(T) = \frac{1}{2\pi}\int_0^{2\pi} F(Te^{i\theta})F_n(e^{i\theta})d\theta,$$

where the integration is again in the Riemann sense.

Thus, by using the properties of Fejér kernel we get that

$$\|\phi_n(F)(T)\| \leq \frac{1}{2\pi}\int_0^{2\pi} \|F(Te^{i\theta})\|F_n(e^{i\theta})d\theta \leq \|F\|_{\mathcal{R}}$$

and hence $\|\phi_n(F)\|_{\mathcal{R}} \leq \|F\|_{\mathcal{R}}$.

The fact that $\phi_n(F)$ converges to F locally uniformly is a standard result for scalar-valued functions. To see it directly in our case note that for $z \in G$, we have that

$$\phi_n(F)(z) = \sum_{|I|\leq n} \frac{(n+1-|I|)}{n+1} A_I z^I = \frac{S_0(z) + \cdots + S_n(z)}{n+1},$$

where $S_k(z) = \sum_{|I|\leq k} A_I z^I, k = 1, \ldots, n$, and hence, $\phi_n(F) \to F$ locally uniformly on G.

For the converse, let $\{\phi_n(F)\}$ be a sequence of $\mathcal{M}_{m,n}$ valued polynomials with $\|\phi_n(F)\|_{\mathcal{R}} \leq 1$ and converging to F pointwise on G. For each n, $\|\phi_n(F)\|_\infty \leq \|\phi_n(F)\|_{\mathcal{R}} \leq 1$. This implies that there exist a subsequence $\{\phi_{n_k}(F)\}$ which converges to a function $H \in \mathcal{M}_{m,n}(H^\infty(G))$ in the weak*-topology and, hence, that $\{\phi_{n_k}(F)\}$ converges to H uniformly on compact subsets of G. Thus, $H = F$ and $\{\phi_{n_k}(F)\}$ converges to F uniformly on compact subsets of G. If we now take $T = (T_1, \ldots, T_N)$ a commuting N-tuple of operators in $\mathcal{Q}(G)$, then by the Taylor functional calculus we have that $\phi_{n_k}(F)(T_1,\ldots,T_N) \longrightarrow F(T_1,\ldots,T_N)$ in norm. Therefore, $\|F(T)\| = \lim_{k\to\infty} \|\phi_{n_k}(F)(T)\| \leq 1$, and hence $\|F\|_{\mathcal{R}} \leq 1$. □

Remark 4.2.5. *The existence of the Fejér kernel gives us a concrete way to approximate elements of $H^\infty_{\mathcal{R}}(G)$ by functions of the algebra of the presentation corresponding to G. In view of Theorem 4.2.2, the above result gives us another way to show that the inclusion of $H^\infty_{\mathcal{R}}(G)$ into $\tilde{\mathcal{A}}$ is isometric.*

The above theorem is the key ingredient in the proof of the GNFT. We now recall another important ingredient, Theorem 3.3.1 — which we actually proved for the algebra of the presentation of *any* analytically presented domain. We would like to remind the reader that the proof of this result does not depend on the theory of operator algebras of functions. Instead it only uses BRS characterization of operator algebras and elementary factorization argument.

Theorem 4.2.6. *Let $G \subseteq \mathbb{C}^N$ be the open unit ball for some norm and assume that G has a finite set of non-constant functions, $\mathcal{R} = \{F_k : G \to M_{m_k,n_k} : 1 \leq k \leq K\}$ as its analytic presentation such that the algebra of the presentation $\mathcal{A}$ is the algebra of polynomials. Let $P = (p_{ij}) \in M_{m,n}(\mathcal{A})$, where $m,\ n$ are arbitrary. Then the following are equivalent:*

(i) $\|P\|_u < 1$,

(ii) *there exist an integer l, matrices of scalars $C_j,\ 1 \leq j \leq l$ with $\|C_j\| < 1$, and admissible block diagonal matrices $D_j(z), 1 \leq j \leq l$, which are of compatible sizes and are such that*

$$P(z) = C_1 D_1(z) \cdots C_l D_l(z).$$

(iii) *there exist a positive, invertible matrix $R \in M_m$, and matrices $P_0, P_k \in M_{m,r_k}(\mathcal{A}), 1 \leq$*

$k \leq m$ such that

$$I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k=1}^{K} P_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)} P_k(w)^*,$$

where $r_k = q_k m_k$ and $z = (z_1, ..., z_N),\ w = (w_1, ..., w_N) \in G$.

We now a give a proof of the GNFT by putting the above two ingredients together.

Theorem 4.2.7. *Let $G \subseteq \mathbb{C}^N$ be the open unit ball for some norm and assume that G has a finite set of non-constant functions, $\mathcal{R} = \{F_k : G \to M_{m_k,n_k} : 1 \leq k \leq K\}$ as its analytic presentation such that the algebra of the presentation $\mathcal{A}$ is the algebra of polynomials and that $e^{i\theta}\mathcal{Q}(G) \subseteq \mathcal{Q}(G)$ for every $\theta \in [0, 2\pi]$. Let $f = (f_{ij})$ be a $M_{m,n}$-valued function defined on G. Then the following are equivalent:*

(1) $f \in M_{mn}(H^\infty_{\mathcal{R}}(G))$ and $\|f\|_{\mathcal{R}} \leq 1$,

(2) there exist an analytic operator-valued function $R_0 : G \to B(\mathcal{H}_0, \mathbb{C}^m)$ and there exist a Hilbert spaces $\mathcal{H}_i$ and an analytic function, $R_i : G \to B(\mathcal{H}_i \otimes \mathbb{C}^m, \mathbb{C}^n)$ such that

$$I_n - f(z)f(w)^* = R_0(z)R_0(w)^* + \sum_{i=1}^{K} R_i(z)[(1 - F_i(z)F_i(w)^*) \otimes I_{H_i}]R_i(w)^*$$

for every $z, w \in G$.

Proof. Assume that $f \in M_{mn}(H^\infty_{\mathcal{R}}(G))$ and $\|f\|_{\mathcal{R}} \leq 1$. By Theorem 4.2.4, there exists a sequence of matrices of polynomials $\{P_n\}$ that converges to f locally uniformly on G with $\|P_n\|_{\mathcal{R}} < 1$ for each n.

By Proposition 4.2.2 and Theorem 4.2.6 there exists a positive, invertible matrix $R^{(n)} \in M_m$ and matrices $P_0^{(n)},\ P_k^{(n)} \in M_{m,r_{k,n}}(\mathcal{A}), 1 \leq k \leq K$ such that

$$I_m - P_n(z)P_n(w)^* = R^{(n)} + P_0^{(n)}(z)P_0^{(n)}(w)^* + \sum_{k=1}^{K} P_k^{(n)}(z)(I - F_k(z)F_k(w)^*)^{(q_{k,n})} P_k^{(n)}(w)^*$$

where $r_{k,n} = q_{k,n} m_k$ and $z = (z_1, ..., z_N),\ w = (w_1, ..., w_N) \in G$.

Now the result follows by using Proposition 3.4.19 and the standard argument of finding the limit points as done in Theorem 3.4.22 . □

Remark 4.2.8. *Since the Examples 3.5.1, 3.5.2, 3.5.3, 3.5.4, and 3.5.7 satisfy the hypotheses of the above theorem, the above result gives us an alternate way to prove GNFT for them.*

4.2.2 Annulus

Our primary aim is to obtain results analogous to Proposition 4.2.2 and Theorem 4.2.4 for the annulus.

First we highlight the basic difference between the annulus and the class of domains that we described in the earlier section. Note that in the case of the annulus there is no natural family of analytic maps of the annulus into itself that play the role that the maps $z \to rz$ played in the proof of Proposition 4.2.2, since an annulus is determined up to biholomorphic equivalence by the ratio of the inner and outer radii. Instead, the proof that $\|\cdot\|_u = \|\cdot\|_{\mathcal{R}}$ is carried out on the level of individual operators. The existence of the sequence of Laurent polynomials that approximates the functions in the $H^\infty_{\mathcal{R}}(\mathbb{A}_r)$ arises from integrating a suitable Fejér-like kernel over the unit circle.

Fix $0 < r < 1$, we define the classical annulus as

$$\mathbb{A}_r = \{z \in \mathbb{C} : r < |z| < 1\} \stackrel{*}{=} \{z \in \mathbb{C} : |z| < 1,\ |z^{-1}| < r^{-1}\}$$

The (*) equality suggests that the annulus is an analytically presented domain with $\mathcal{R} = \{F_1, F_2 : F_1(z) = z, F_2(z) = rz^{-1}\}$ as its analytic presentation and the algebra of the presentation, $\mathcal{A}$ is the algebra of Laurent polynomials.

Thus, the quantized version of the annulus is given by

$$\mathcal{Q}(\mathbb{A}_r) = \{T \in B(H) : \|T\| \leq 1,\ \|T^{-1}\| \leq r^{-1},\ \sigma(T) \subseteq \mathbb{A}_r\}.$$

An admissible representation is given by an invertible operator T on some Hilbert space $\mathcal{H}$ that satisfies $\|T\| \leq 1$ and $\|T^{-1}\| \leq r^{-1}$ and a strict admissible representation is given by an invertible operator S on some Hilbert space that satisfies the strict inequalities, that is, $\|S\| < 1$ and $\|S^{-1}\| < r^{-1}$.

As we saw in the case of the unit ball in $\mathbb{C}^N$, we have three possible norm structures on the algebra of the presentation. In the next proposition, we prove that they are equal. Note that the annulus is a circular domain and $e^{i\theta}\mathcal{Q}(\mathbb{A}_r) \subseteq \mathcal{Q}(\mathbb{A}_r)$ for every $\theta \in [0, 2\pi]$, but still the proof of Theorem 4.2.2 fails in this case. Thus, we employ other techniques to obtain the similar result.

Proposition 4.2.9. *Let $\mathcal{A}$ be the algebra of the presentation generated by the above defined functions F_1 and F_2. Then, $\mathcal{A} = \mathcal{A}_0$ completely isometrically and the inclusion of $\mathcal{A}$ into $H^\infty_{\mathcal{R}}(G)$ is isometric.*

Proof. It is easy to see that every strict admissible representation is given by an invertible operator T that belongs to $\mathcal{Q}(\mathbb{A}_r)$ and every operator $T \in \mathcal{Q}(\mathbb{A}_r)$ gives rise to an admissible representation π such that $\pi(z) = T$. Thus, $\|f\|_{u_0} \leq \|f\|_{\mathcal{R}} \leq \|f\|_u$ for every $f \in \mathcal{A}$.

Let π be an admissible representation such that $\pi(z) = T$, then $\|T\| \leq 1$ and $\|T^{-1}\| \leq r^{-1}$. Since T is invertible, thus by polar decomposition, there exists an unitary U and a positive operator P such that $T = UP$. The fact that P is positive together with $\|T\| \leq 1$ and $\|T^{-1}\| \leq r^{-1}$ implies that $r \leq P \leq 1$.

Let $\epsilon > 0$, take $P_\epsilon = \frac{P+\epsilon(1+r)}{1+2\epsilon}$. Then obviously P_ϵ is a positive operator for every $\epsilon > 0$.

It is easy to see that the spectrum of P_ϵ satisfies

$$\sigma(P_\epsilon) \subseteq \left[\frac{r+\epsilon(1+r)}{1+2\epsilon}, \frac{1+\epsilon(1+r)}{1+2\epsilon}\right].$$

The property that $r < 1$ and a simple calculation shows that

$$\left[\frac{r+\epsilon(1+r)}{1+2\epsilon}, \frac{1+\epsilon(1+r)}{1+2\epsilon}\right] \subseteq (r,1).$$

This shows that $r < P_\epsilon < 1$ and thus, we define $T_\epsilon = UP_\epsilon$ which satisfies the required inequalities: $\|T_\epsilon\| < 1$ and $\|T_\epsilon^{-1}\| < r^{-1}$.

Let $f \in \mathcal{A}$, so that $f(z) = \sum_{k=-d}^{d} \alpha_k z^k$. Then

$$\|f(T) - f(T_\epsilon)\| \leq \sum_{k=-d}^{d} |\alpha_k| \|(UP)^k - (UP_\epsilon)^k\|.$$

Note that

$$\|(UP)^k - (UP_\epsilon)^k\| \leq \begin{cases} k\|P - P_\epsilon\| & \text{if } k > 0, \\ -k\|P^{-1} - P_\epsilon^{-1}\| & \text{if } k < 0. \end{cases}$$

Since $\|P_\epsilon - P\| \to 0$ and $\|P_\epsilon^{-1} - P^{-1}\| \to 0$ as $\epsilon \to 0$, we have that $\lim_{\epsilon\to 0} \|f(T) - f(T_\epsilon)\| = 0$. This shows that

$$\|f(T)\| = \lim_{\epsilon\to 0} \|f(T_\epsilon)\| \leq \{\|f(S)\| : \|S\| < 1, \|S^{-1}\| < r^{-1}\} = \|f\|_{u_0}$$

for every T such that $\|T\| \leq 1$ and $\|T^{-1}\| \leq r^{-1}$. Thus, we obtain that $\|f\|_{u_0} = \|f\|_{\mathcal{R}} = \|f\|_u$ for every $f \in \mathcal{A}$. This proves that $\mathcal{A} = \mathcal{A}_0$ isometrically and also the inclusion of $\mathcal{A}$ into $H^\infty_R(\mathbb{A}_r)$ is isometric.

The complete proof of the result can be obtained by following the exact same proof as above for the matrices in $\mathcal{A}$. □

Let $f : \mathbb{A}_r \to \mathbb{C}$ be an analytic function then the Laurent series expansion of f is given by $f(z) = \sum_{k=-\infty}^{\infty} \alpha_k z^k$. We can write $f(z) = f_1(z) + f_2(z)$ where $f_1(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ is analytic over $\{z : |z| < 1\}$ and $f_2(z) = \sum_{k=1}^{\infty} \alpha_{-k} z^{-k}$ is analytic over $\{z : |z^{-1}| < r^{-1}\}$. Thus, the natural way to obtain approximating Laurent polynomials is by integrating f_1 and f_2 against respective Fejér kernels which can be obtained by similar argument as in the case of the unit ball. Instead, we find a different Fejér-like kernel to obtain an approximating sequence of Laurent polynomials.

Fix an $n \in \mathbb{N}$ and we define the Fejér-like kernel,

$$F_n'(\theta) = F_{2n}(\theta),$$

where $F_n(theta) = \frac{1}{n+1} \sum_{k,l=0}^{n} e^{i(k-l)\theta}$ for every $\theta \in [0, 2\pi]$, as defined in the last section. Thus, it is easy to see that $F_n'(\theta) \geq 0$, $\frac{1}{2\pi} \int_0^{2\pi} F_n'(\theta) = 1$, and $F_n'(\theta) = \frac{1}{2n+1} \sum_{k,l=-n}^{n} e^{i(k-l)\theta} = \sum_{k=-2n}^{2n} (1 - \frac{|k|}{2n+1}) e^{ik\theta}$.

Theorem 4.2.10. *Let $F \in \mathcal{M}_{m,n}(H^\infty_{\mathcal{R}}(\mathbb{A}_r))$ with $F(z) = \sum_{i=-\infty}^{\infty} A_i z^i$ for $z \in \mathbb{A}_r$. Then the sequence $\{P_n\}$ of matrices of Laurent polynomials $P_n(z) = \sum_{|i| \leq 2n} (1 - \frac{|i|}{2n+1}) A_i z^i$ converges locally uniformly to F and $\|P_n\|_{\mathcal{R}} \leq \|F\|_{\mathcal{R}}$ for each n. Conversely, if there is a sequence of P_n, matrices of Laurent polynomials, converging to F pointwise on $\mathbb{A}_r$ with $\|P_n\|_{\mathcal{R}} \leq 1$ for each n, then $\|F\|_{\mathcal{R}} \leq 1$.*

Proof. Fix an $n \in \mathbb{N}$ and consider the Fejér-like kernel,

$$F_n'(\theta) = \frac{1}{2n+1} \sum_{k,l=-n}^{n} e^{i(k-l)\theta} \text{ for } \theta \in [0, 2\pi].$$

Note that for each fixed $z \in \mathbb{A}_r$ the function $\theta \mapsto F(ze^{i\theta}) = F(z_1 e^{i\theta}, \ldots, z_N e^{i\theta})$ is continuous. We define $P_n(z) = \frac{1}{2\pi} \int_0^{2\pi} F(ze^{i\theta}) F_n'(e^{i\theta}) d\theta \quad$ for $z \in \mathbb{A}_r$, where the integration is in the Riemann sense. A direct calculation shows that $P_n(z) = \sum_{|i| \leq 2n} (1 - \frac{|i|}{2n+1}) A_i z^i$.

Next check that for a fixed commuting N-tuple of operators, $T = (T_1, \ldots, T_N) \in \mathcal{Q}(G)$, on a Hilbert space $\mathcal{H}$, the map $\theta \mapsto F(T_1e^{i\theta}, \ldots, T_Ne^{i\theta})$ is continuous from the interval into $B(\mathcal{H})$ equipped with the norm topology. This follows from the fact that $\sigma(e^{i\theta}T) \subset G$, $F(Te^{i\theta})$ is a norm limit of partial sums of its Laurent series. It now follows that

$$P_n(T) = \tfrac{1}{2\pi} \int_0^{2\pi} F(Te^{i\theta})F_n'(e^{i\theta})d\theta,$$

where the integration is again in the Riemann sense.

Thus, by using the properties of the Fejér kernel we get that

$$\|P_n(T)\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|F(Te^{i\theta})\| F_n'(e^{i\theta})d\theta \leq \|F\|_{\mathcal{R}}$$

and we have shown that $\|P_n\|_{\mathcal{R}} \leq \|F\|_{\mathcal{R}}$.

The fact that P_n converges to F locally uniformly is a standard result for scalar-valued functions. To see it directly in our case note that for $z \in \mathbb{A}_r$, we have that

$$P_n(z) = \sum_{|i| \leq 2n} \frac{(2n+1-|i|)}{2n+1} A_i z^i = \frac{S_0(z) + \cdots + S_{2n}(z)}{2n+1},$$

where $S_k(z) = \sum_{|i| \leq k} A_i z^i$, and hence, $P_n \to F$ locally uniformly on $\mathbb{A}_r$.

For the converse, let $\{P_n\}$ be a sequence of $\mathcal{M}_{m,n}$ valued polynomials with $\|P_n\|_{\mathcal{R}} \leq 1$ and converging to F pointwise on $\mathbb{A}_r$. For each n, $\|P_n\|_\infty \leq \|P_n\|_{\mathcal{R}} \leq 1$. This implies that there exists a subsequence $\{P_{n_k}\}$ which converges to a function $H \in \mathcal{M}_{m,n}(H^\infty(\mathbb{A}_r))$ in the weak*-topology. This further implies that $\{P_{n_k}\}$ converges to H uniformly on compact subsets of $\mathbb{A}_r$. Thus, $H = F$ and $\{P_{n_k}\}$ converges to F uniformly on compact subsets of $\mathbb{A}_r$. If we now take $T = (T_1, \ldots, T_N)$ a commuting N-tuple of operators in $\mathcal{Q}(\mathbb{A}_r)$, then by the Taylor functional calculus we have that $P_{n_k}(T_1, \ldots, T_N) \longrightarrow F(T_1, \ldots, T_N)$ in norm. Therefore, $\|F(T)\| = \lim_{k\to\infty} \|P_{n_k}(T)\| \leq 1$ and hence, $\|F\|_{\mathcal{R}} \leq 1$. $\square$

Remark 4.2.11. *In the above theorem, for every function $f \in H^\infty_{\mathcal{R}}(G)$ we have constructed a sequence of Laurent polynomials that approximates f. Thus, this gives us another way to prove that $H^\infty_{\mathcal{R}}(G) \subseteq \tilde{\mathcal{A}}$ completely isometrically.*

In the view of the above theorem, we can obtain GNFT for the annulus using Proposition 4.2.9 and Theorem 3.3.1 as in the case of the unit ball. We study this example in more detail in the next chapter where we sketch another proof of the GNFT for the annulus using Agler's factorization result.

Chapter 5

Case Study of the Quantum Annulus

5.1 Introduction

This chapter is dedicated to the study of the quantum analogue of one of the important domains in classical complex analysis.

For fixed $0 < R_1 < R_2$, the annulus is defined by the set

$$\mathbb{A} = \{z \in \mathbb{C} : R_1 < |z| < R_2\}.$$

By using the standard fact about the annulus, that two annuli with the same inner and outer radius ratio are conformally equivalent (i.e., there exists a bijective analytic map between them), we get that the annulus with inner radius R_1 and outer radius R_2 is conformally equivalent to the annulus with inner radius $r = \frac{R_1}{R_2}$ and outer radius 1. Thus, it is enough to lead the discussion by restricting to an annulus that has inner radius equal to r and a fixed outer radius 1.

For fixed $0 < r < 1$, let

$$\mathbb{A}_r = \{z \in \mathbb{C} : r < |z| < 1\}$$

denote the annulus with inner radius r and outer radius 1. As illustrated in Chapter 3, we will consider a quantized version of an annulus which is defined via the set

$$\mathbb{Q}(\mathbb{A}_r) = \{T : \|T\| \le 1, \|T^{-1}\| \le r^{-1},\ \sigma(T) \subseteq \mathbb{A}_r\}.$$

We refer to this set as the *quantum annulus*.

The space of all bounded analytic functions on $\mathbb{A}_r$ is denoted by

$$H^\infty(\mathbb{A}_r) = \{f : \mathbb{A}_r \to \mathbb{C} : f \text{ is analytic and } \|f\|_\infty < \infty\}$$

where $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{A}_r\}$.

We denote the algebra of Laurent polynomials by $\mathbb{P}_{\mathcal{L}}$. Recall from the example section of Chapter 3 that $\mathbb{A}_r$ is an analytically presented domain and the algebra of the presentation turns out to be the algebra of Laurent polynomials. The space of analytic functions bounded on the quantum annulus is denoted by

$$H^\infty_{\mathcal{R}}(\mathbb{A}_r) = \{f : \mathbb{A}_r \to \mathbb{C} : f \text{ is analytic and } \|f\|_{\mathcal{R}} < \infty\}$$

where $\|f\|_{\mathcal{R}} = \sup\{\|f(T)\| : T \in \mathcal{Q}(\mathbb{A}_r)\}$.

For a fixed constant $K \ge 1$, a closed subset X of the complex plane which contains the spectrum $\sigma(T)$ is called a K-spectral set for T if the inequality

$$\|f(T)\| \le K\|f\|_X, \text{ where } \|f\|_X := \sup_{x \in X} |f(x)|,$$

holds for all complex-valued bounded rational functions on X, and K is called a spectral constant. Furthermore, if $K = 1$, the set X is said to be a spectral set for T. The set X

is said to be a complete K-spectral for T if the inequality

$$\|F(T)\| \leq K\|F\|_X, \text{ where } \|F\|_X = \sup_{x\in X} \|F(x)\|,$$

holds for all $n \times n$ matrix-valued bounded rational functions F defined on X, and for all values of n. The constant K is called a complete spectral constant. In the case $K = 1$, the set X is said to be a complete spectral set for T. We refer to the expository article by Paulsen which appeared in [59], and to the book [62] for modern surveys of known properties of K-spectral and complete K-spectral sets.

It was shown long ago that the annulus is a K-spectral set with $K > 1$ for any operator $T \in \mathcal{Q}(\mathbb{A}_r)$. In fact, it is a complete K-spectral set. The problem of finding the smallest C such that the annulus is a C-spectral set or complete C-spectral set for all $T \in \mathcal{Q}(\mathbb{A}_r)$ is a long standing problem. Unfortunately, no satisfactory result is known yet. We attempt to tackle this problem via two different approaches; a concrete and an abstract approach. A concrete approach uses pseudohyperbolic distance and an abstract approach uses the theory of hyperconvex sets [27]. We illustrate these ideas in Section 5.6.

This chapter is organized as follows. In Section 5.2, we derive another proof of GNFT and GNPP by embedding the annulus into the bidisk. This will be the third proof of GNFT for the annulus presented in this thesis, including the one in Chapter 3 and the other in Chapter 4. In the next section, we define an appropriate quantum analogue of pseudohyperbolic distance of points in the annulus, and we use this to obtain the solution of the two-point interpolation problem for the quantum annulus in terms of the distance formula, as in the case of the solution of the two-point classical Nevanlinna-Pick interpolation problem. Finally, in the last section we employ two different approaches to find an estimate of the lower bound of the optimal spectral constant.

5.2 GNFT and GNPP

We embed the annulus $\mathbb{A}_r = \{z : r < |z| < 1\}$ into the bidisk $\mathbb{D}^2 = \{(z_1, z_2) : |z_1| < 1,\ |z_2| < 1\}$ via the natural embedding map $\gamma : \mathbb{A}_r \to \mathbb{D}^2$ which is defined as $\gamma(z) = (z, rz^{-1})$. Let $\mathbb{P}_2$ denote the algebra of polynomials in two variables. Let $I =< z_1 z_2 - r >$ be the ideal generated by $\{z_1 z_2 - r\}$ and let V be the algebraic set defined as $V := Z(I) := \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_1 \lambda_2 - r = 0\}$. Note that the ideal can be written as

$$I = \{p \in \mathbb{P}_2 : p|_V = 0\} = \{p \in \mathbb{P}_2 : p(\lambda, r\lambda^{-1}) = 0 \ \forall \ \lambda \in \mathbb{C} \setminus \{0\}\}.$$

We may extend the map γ to the whole of $\mathbb{C} \setminus \{0\}$ via the same definition,

$$\gamma : \mathbb{C} \setminus \{0\} \to \mathbb{C}^2, \quad \gamma(z) = (z, rz^{-1}).$$

This allows us to define a map $\gamma^* : \mathbb{P}_2 \to \mathbb{P}_{\mathcal{L}}$ such that $\gamma^*(p) := p \circ \gamma = p(z, rz^{-1})$ for every $p \in \mathbb{P}_2$. It is easy to see that this map is onto and the kernel of this map, $Ker(\gamma^*) = I$. Thus, this map induces a matrix-norm structure on $\mathbb{P}_{\mathcal{L}}$ by using the matrix-norm structure on $\mathbb{P}_2$ and the identification $\frac{\mathbb{P}_2}{I} \cong \mathbb{P}_{\mathcal{L}}$. Let $(p_{ij}) \in M_n(\mathbb{P}_{\mathcal{L}})$, then

$$\|(p_{ij})\|_q = \inf\{\|(h_{ij})\|_\infty : \gamma^*(h_{ij}) = p_{ij}\}$$

where $\|(h_{ij})\|_\infty = \sup\{\|(h_{ij}(z))\| : z \in \mathbb{D}^2\}$. It then follows from the main theorem of [25] that $\mathbb{P}_{\mathcal{L}}$ satisfies the axioms to be an operator algebra. We now recall other norm structures on $M_n(\mathbb{P}_{\mathcal{L}})$ which comes from the fact that it is an algebra of the presentation of the annulus. Let $(p_{ij}) \in M_n(\mathbb{P}_{\mathcal{L}})$, then

$$\|(p_{ij})\|_u = \{\|(p_{ij}(T))\| : \|T\| \le 1, \|T^{-1}\| \le r^{-1}\}$$

and

$$\|(p_{ij})\|_{u_0} = \{\|(p_{ij}(T))\| : \|T\| < 1, \|T^{-1}\| < r^{-1}\}.$$

Proposition 5.2.1. *Let $P = (p_{ij}) \in M_n(\mathbb{P}_{\mathcal{L}})$, then $\|P\|_u = \|P\|_{u_0} = \|P\|_q$.*

Proof. It was shown in Proposition 4.2.9 that $\|P\|_u = \|P\|_{u_0}$ for every $P \in M_n(\mathbb{P}_{\mathcal{L}})$, and for every n. Suppose $\|P\|_q < 1$, then there exists $H = (h_{ij}) \in M_n(\mathbb{P}_2)$ such that $\|H\|_\infty < 1$ and $\gamma^*(h_{ij}) = p_{ij}$ for every i, j. Hence, for $T \in B(\mathcal{H})$ which satisfy $\|T\| \leq 1$ and $\|rT^{-1}\| \leq 1$, we have $p_{ij}(T) = h_{ij}(T, rT^{-1})$. But $A = T$ and $B = rT^{-1}$ are commuting contractions so that

$$\|(p_{ij}(T))\| = \|(h_{ij}(A, B))\| \overset{*}{\leq} \|(h_{ij})\|_\infty < 1$$

where the inequality (*) follows from Ando's inequality. By taking the supremum over all $T \in B(\mathcal{H})$ with $\|T\| \leq 1$, $\|rT^{-1}\| \leq 1$ and Hilbert space $\mathcal{H}$, we get that $\|P\|_u \leq 1$. This proves that $\|P\|_u \leq \|P\|_q$.

Since $(\mathbb{P}_{\mathcal{L}}, \|.\|_q)$ is an operator algebra, it follows by Theorem 2.1.1 that there exist a Hilbert space $\mathcal{H}$ and an algebra homomorphism $\pi : \mathbb{P}_{\mathcal{L}} \to B(\mathcal{H})$ such that $\|(\pi(a_{ij}))\| = \|(a_{ij})\|_q$ for every $(a_{ij}) \in M_n(\mathbb{P}_{\mathcal{L}})$, and for every n. Note $\pi(z) = T$, then we have that

$$\|T\| = \inf\{\|h\|_\infty : h(z, rz^{-1}) = z\} \leq \|z_1\|_\infty \leq 1$$

and

$$\|rT^{-1}\| = \inf\{\|h\|_\infty : h(z, rz^{-1}) = rz^{-1}\} \leq \|z_2\|_\infty \leq 1.$$

This further implies that

$$\|(p_{ij})\|_q = \|(\pi(p_{ij}))\| = \|(p_{ij}(T))\| \leq \|(p_{ij})\|_u$$

and hence completes the proof of the result.

□

We now wish to define a norm on the bounded analytic functions on the annulus, $H^\infty(\mathbb{A}_r)$, using the embedding map γ. It is easy to see that the map $\gamma^* : H^\infty(\mathbb{D}^2) \to$

$H^\infty(\mathbb{A}_r)$ defined via $\gamma^*(f) = f \circ \gamma$ is a well-defined homomorphism with $Ker(\gamma^*) = \{g \in H^\infty(\mathbb{D}^2) : g(\lambda, r\lambda^{-1}) = 0 \ \forall\ \lambda \in \mathbb{A}_r\}$.

Proposition 5.2.2. *The above defined map $\gamma^* : H^\infty(\mathbb{D}^2) \to H^\infty(\mathbb{A}_r)$ is an onto map.*

Proof. Let $f \in H^\infty(\mathbb{A}_r)$, then $f(z) = \sum_{j=-\infty}^{\infty} a_j z^j$ such that $f_+(z) = \sum_{j=0}^{\infty} a_j z^j$ and $f_-(z) = \sum_{j=-\infty}^{-1} a_j z^j$ satisfy $\|f_+\|_{\infty,\mathbb{A}_r} \leq C_1(r)\|f\|_{\infty,\mathbb{A}_r}$ and $\|f_-\|_{\infty,\mathbb{A}_r} \leq C_2(r)\|f\|_{\infty,\mathbb{A}_r}$ where $C_1(r)$ and $C_2(r)$ are constants that depends on r, see [74]. If we define $g_+(z_1,z_2) := f_+(z_1)$, then clearly g_+ is an analytic function on $\mathbb{D}^2$ such that $\|g_+\|_{\infty,\mathbb{D}^2} \leq C_1(r)\|f\|_{\infty,\mathbb{A}_r}$. Clearly, if we let $g_-(z_1,z_2) := f_-(r/z_2) = \sum_{j=1}^{\infty} a_{-j}(z_2/r)^j$, then g_- is an analytic function on $\mathbb{D}^2$. To see that $g_- \in H^\infty(\mathbb{D}^2)$, consider

$$\begin{aligned}\|g_-\|_{\infty,\mathbb{D}^2} &= \sup\{|f_-(r/z_2)| : |z_2| \leq 1\} = \sup\{|\sum_{j=1}^{\infty} a_{-j}w^j| : |w| \leq r^{-1}\} \\ &\overset{*}{=} \sup\{|\sum_{j=1}^{\infty} a_{-j}w^j| : |w| = r^{-1}\} = \sup\{|\sum_{j=1}^{\infty} a_{-j}z^{-j}| : |z| = r\} = \|f_-\|_{\infty,\mathbb{A}_r} \leq C_1(r)\|\end{aligned}$$

where the equality (*) follows from the maximum modulus theorem. Thus, if we define $g(z_1,z_2) = g_+(z_1,z_2) + g_-(z_1,z_2)$, then $g \in H^\infty(\mathbb{D}^2)$ satisfies $\gamma^*(g) = f$ and hence completes the proof of the result. □

Define a norm on $M_k(H^\infty(\mathbb{A}_r))$ using the identification $\frac{H^\infty(\mathbb{D}^2)}{Ker(\gamma^*)} \cong H^\infty(\mathbb{A}_r)$. Given $F = (f_{ij}) \in M_k(H^\infty(\mathbb{A}_r))$, we have that $\|F\|_{q'} = \inf\{\|(h_{ij})\|_\infty : \gamma^*(h_{ij}) = f_{ij} \ \forall\ i,\ j\}$. Recall, there is another norm structure on $M_k(H^\infty(\mathbb{A}_r))$,

$$\|F\|_{\mathcal{R}} = \sup\{\|F(T)\| : T \in \mathcal{Q}(\mathbb{A}_r)\} = \sup\{\|F(T)\| : \|T\| \leq 1,\ \|T^{-1}\| \leq r^{-1} \text{ and } \sigma(T) \subseteq \mathbb{A}_r\}.$$

Theorem 5.2.3. *Let $F = (f_{ij}) \in M_k(H^\infty(A_r))$. Then $\|F\|_{q'} < 1$ if and only if there exists $P_n \in M_k(\mathbb{P}_{\mathcal{L}})$ such that $P_n(z) \to F(z)$ for every $z \in \mathbb{A}_r$ and $\|P_n\|_q < 1$. Consequently, $\|F\|_{q'} = \|F\|_{\mathcal{R}}$.*

Proof. Assume $\|F\|_{q'} < 1$, then this implies that there exists $H = (h_{ij}) \in M_k(H^\infty(\mathbb{D}^2))$ such that $f_{ij}(z) = h_{ij}(z, rz^{-1})$ and $\|H\|_{\infty,\mathbb{D}^2} < 1$. Since $\|H\|_{\infty,\mathbb{D}^2} < 1$, there exists a sequence of matrices of polynomials G_n such that $G_n \to H$ pointwise on $\mathbb{D}^2$ and $\|G_n\|_{\infty,\mathbb{D}^2} < 1$. Let $P_n(z) = G_n(z, rz^{-1})$, then $P_n \to F$ pointwise on $\mathbb{A}_r$ and $\|P_n\|_q \leq \|G\|_{\infty,\mathbb{D}^2} < 1$. Conversely, suppose there exists $P_n \in M_k(\mathbb{P}_{\mathcal{L}})$ such that $P_n(z) \to F(z)$ for every $z \in \mathbb{A}_r$ and $\|P_n\|_q < 1$. Note that $\|P_n\|_q < 1$ implies that there exists a sequence of matrices of polynomials G_n such that $G_n(z, rz^{-1}) = P_n(z)$ and $\|G_n\|_{\infty,\mathbb{D}^2} < 1$. A standard fact that the unit ball of $H^\infty(\mathbb{D}^2)$ is weak*-compact implies that there exists $G \in H^\infty(D^2)$, a weak*-limit of G_n. Thus, we have that $G(z, rz^{-1}) = F(z)$ for every $z \in \mathbb{A}_r$ and $\|G\|_{\infty,\mathbb{D}^2} < 1$. This proves that $\|F\|_{q'} < 1$. Note that the last statement of the result follows from Theorem 4.2.10. □

Corollary 5.2.4. *For $P \in M_k(\mathbb{P}_{\mathcal{L}})$, $\|P\|_q = \|P\|_{q'}$.*

Lemma 5.2.5. *Let $F \in M_k(H^\infty(\mathbb{A}_r))$ with $\|F\|_{q'} = 1$, then there exists $H \in M_k(H^\infty(\mathbb{D}^2))$ such that $\gamma^{*^{(k)}}(H) = F$ and $\|H\|_\infty = 1$.*

Proof. By the definition of the $\|.\|_q$ norm there exists a sequence $H_n \in M_k(H^\infty(\mathbb{D}^2))$ such that $\gamma^*(H_n) = F$ and $\|H_n\|_\infty \leq (1 + \frac{1}{n})$. Let H be a weak*-limit point of H_n. Then $\|H\|_\infty \leq 1$, and $H(z, rz^{-1}) = F(z)$. □

The above lemma allows us to present another proof of the generalized Nevanlinna Factorization theorem.

Theorem 5.2.6. *Let $F \in M_k(H^\infty(\mathbb{A}_r))$. Then $\|F\|_{q'} \leq 1$ if and only if there exist positive definite functions P, Q on $\mathbb{A}_r$ such that*

$$I - F(z)F(w)^* = (1 - z\overline{w})P(z,w) + (1 - r^2 z^{-1}\overline{w}^{-1})Q(z,w)$$

holds for every $z, w \in \mathbb{A}_r$.

Proof. First, we assume that $\|F\|_{q'} \leq 1$. Then pick $H \in M_k(H^\infty(\mathbb{D}^2))$, such that $H(z, rz^{-1}) = F(z)$ for every $z \in \mathbb{A}_r$ and $\|H\|_\infty = \|F\|_{q'} \leq 1$. By Theorem 3.4.22 or by Agler's factorization result [3], we get that there exist two positive definite function P_1, Q_1 on $\mathbb{D}^2$ such that

$$I - H(z_1, z_2)H(w_1, w_2)^* = (1 - z_1\overline{w_1})P_1(z_1, z_2, w_1, w_2) + (1 - z_2\overline{w_2})Q_1(z_1, z_2, w_1, w_2)$$

holds for every $(z_1, z_2), (w_1, w_2) \in \mathbb{D}^2$. We define $P : \mathbb{A}_r \times \mathbb{A}_r \to M_k$ and $Q : \mathbb{A}_r \times \mathbb{A}_r \to M_k$ via the maps $P(z, w) = P_1(z, rz^{-1}, w, rw^{-1})$ and $Q(z, w) = Q_1(z, rz^{-1}, w, rw^{-1})$ respectively. Note that each of P and Q are positive definite functions on $\mathbb{A}_r \times \mathbb{A}_r$ such that

$$I - F(z)F(w)^* = (1 - z\overline{w})P(z, w) + (1 - r^2 z^{-1}\overline{w}^{-1})Q(z, w)$$

holds for every $z, w \in \mathbb{A}_r$.

To show the reverse implication, we assume that there exist positive definite functions P, Q on $\mathbb{A}_r \times \mathbb{A}_r$ such that

$$I - F(z)F(w)^* = (1 - z\overline{w})P(z, w) + (1 - r^2 z^{-1}\overline{w}^{-1})Q(z, w)$$

holds for every $z, w \in \mathbb{A}_r$. By using Theorem 1.4.4, we can factor $P,\ Q$ such that $P(z, w) = P'(z)P'^*(w)$ and $Q(z, w) = Q'(z)Q'^*(w)$. Fix $T \in \mathcal{Q}(\mathbb{A}_r)$, then $I - F(T)F(T)^* \geq 0$ which further implies that $\|F(T)\| \leq 1$ for every $T \in Q(G)$. By Theorem 5.2.3, we get that $\|F\|_q = \|F\|_{\mathcal{R}} \leq 1$ and hence it completes the proof. □

The following theorem gives us the solution of the generalized Nevanlinna-Pick interpolation problem for the annulus.

Theorem 5.2.7. *Let $\{z_1, z_2, ..., z_n\} \subseteq \mathbb{A}_r$ and $\{W_1, ..., W_n\} \in M_k$. Then there exists $F \in M_k(H^\infty(\mathbb{A}_r))$ such that $F(z_i) = W_i$ for every i and $\|F\|_{q'} \leq 1$ if and only if there exists* *positive definite matrices* $(A_{ij}), (B_{ij}) \in M_n(M_k)$ *such that*

$$I - W_i W_j^* = (1 - z_i \overline{z_j})A_{ij} + (1 - r^2 {z_i}^{-1} \overline{z_j}^{-1})B_{ij}$$

for every $1 \leq i, j \leq n$.

Proof. By using the same idea as in the above theorem and the solution of the Nevanlinna-Pick interpolation problem for the bidisk [2], we can easily get the proof of this result. We leave the details for the reader to verify. □

5.3 Distance Formulae

We begin this section by recalling some of the important notions from geometric function theory. We shall use $Hol(X, \mathbb{D})$ to denote the set of holomorphic functions from X to the unit disk $\mathbb{D}$. Given two points λ_1 and λ_2 in $\mathbb{D}$ the *pseudohyperbolic distance* between them is defined to be

$$\rho(\lambda_1, \lambda_2) = \left| \frac{\lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2} \right| = \phi_{\lambda_2}(\lambda_1)$$

where $\phi_a(z) := \frac{z-a}{1-\bar{a}z}$ defines an automorphism of the unit disk $\mathbb{D}$ for every $a \in \mathbb{D}$. These maps are often called *Möbius transformations.*

The following variant of the Schwarz lemma after Pick's result [66] is known as the Schwarz-Pick lemma [70]. The statement is as follows:

Theorem 5.3.1 (Schwarz-Pick). *Suppose* $\lambda_1, \lambda_2, w_1, w_2$ *are points in* $\mathbb{D}$. *Then there exists a holomorphic function* $\phi : \mathbb{D} \to \mathbb{D}$ *that maps* λ_1 *to* w_1 *and* λ_2 *to* w_2 *if and only if*

$$\rho(w_1, w_2) \leq \rho(\lambda_1, \lambda_2).$$

This theorem implies that among all holomorphic maps from $\mathbb{D}$ to $\mathbb{D}$ that vanish at λ_1, the Möbius map ϕ_{λ_1} has the maximum modulus at λ_2,

$$\rho(\lambda_1, \lambda_2) = \sup\{|f(\lambda_2)| : f \in Hol(\mathbb{D}, \mathbb{D}),\ f(\lambda_1) = 0\}.$$

A generalized notion of pseudodistance arises from the above observation. This gives rise to a notion of pseudodistance on any domain $G \subseteq \mathbb{C}^N$ which Jarnicki and Pflug [47] call the *Möbius pseudodistance:*

$$\rho(z_1, z_2) = \sup\{|f(z_2)| : f \in Hol(G, \mathbb{D}),\ f(z_1) = 0\}.$$

For this pseudodistance, Jarnicki and Pflug [47] have proved an analogue of the Schwarz-Pick lemma for any domain $G \subseteq \mathbb{C}^N$ which they call the *general Schwarz-Pick lemma*, see Theorem 2.1.1[47].

This notion of pseudodistance was further generalized to *Gleason distance*, see [20]. Let $C(X)$ be the set of all bounded continuous functions on a compact set X and let $\mathcal{A} \subset C(X)$ be a uniform algebra equipped with the sup norm. Let p, q be two points in X. Then the Gleason distance between p and q for the uniform algebra $\mathcal{A}$ is defined to be

$$d_{\mathcal{A}}(p, q) = \sup\{|f(p)| : f \in \mathcal{A},\ \|f\|_\infty < 1, f(q) = 0\}.$$

Note that when $X = \mathbb{D}$ and $\mathcal{A} = H^\infty(\mathbb{D})$, $d_{\mathcal{A}}(p, q) = \rho(p, q)$.

The definition of $d_{\mathcal{A}}(p, q)$ can be formally extended to the case where $\mathcal{A}$ is an operator algebra of functions on some set X. Let $\mathcal{A}$ be an operator algebra of functions on some set X, then an appropriate analogue of the Gleason distance is given by

$$d_{\mathcal{A}}(p, q) := \sup\{|f(p)| : f \in \mathcal{A},\ \|f\|_{\mathcal{A}} < 1, f(q) = 0\}.$$

Recall, the inclusion of every operator algebra of functions into $\ell^\infty(X)$ is completely contractive. This guarantees that the above supremum is finite. We call this the *generalized* *pseudohyperbolic distance*, which we abbreviate as GPHD. Since we are dealing with operator algebras, thus it is natural to introduce a notion of *"complete" GPHD*. We call $d^n_{\mathcal{A}}(p,q)$ *n-GPHD* if

$$d^n_{\mathcal{A}}(p,q) := \sup\{\|(f_{ij}(p))\| : (f_{ij}) \in M_n(\mathcal{A}),\ (f_{ij}(q)) = 0,\ \|(f_{ij})\|_{M_n(\mathcal{A})} < 1\}$$

and we call $d^{cb}_{\mathcal{A}}(p,q)$ *"complete" GPHD* if $d^{cb}_{\mathcal{A}}(p,q) := \sup_n d^n_{\mathcal{A}}(p,q)$. Note that $d^1_{\mathcal{A}}(p,q) = d_{\mathcal{A}}(p,q)$ for every $p,q \in X$. Obviously, $0 \leq d_{\mathcal{A}}(p,q) \leq d^n_{\mathcal{A}}(p,q) \leq 1$ for every $p,q \in X$ and for every $n \in \mathbb{N}$.

Proposition 5.3.2. *Let $\mathcal{A}$ be an operator algebra of functions on the set X and let $d_{\mathcal{A}}(p,q)$ and $d^n_{\mathcal{A}}(p,q)$ be as defined above. Then $d^n_{\mathcal{A}}(p,q) \leq d_{\mathcal{A}}(p,q)$ for every n and for every $p,q \in X$. Consequently, $d_{\mathcal{A}}(p,q) = d^n_{\mathcal{A}}(p,q) = d^{cb}_{\mathcal{A}}(p,q)$ for every $p,q \in X$ and for every n.*

Proof. Fix $p,q \in X$ and n. Let $F = (f_{ij}) \in M_n(\mathcal{A})$ be such that $F(q) = 0$ and $\|F\|_{M_n(\mathcal{A})} \leq 1$. By using the fact that the unit ball of finite dimensional normed space is compact, we find that there exists $v, w \in C^n$ with $\|v\| = \|w\| = 1$ so that $\|F(p)\| = |\langle F(p)v, w\rangle|$.

If we define $f(z) = \langle F(z)v, w\rangle$ for every $z \in X$, then it is easy to see that $f \in \mathcal{A}$ and it is straightforward from the definition of f that $f(q) = 0$ and $|f(p)| = \|F(p)\|$.

Since $\mathcal{A}$ is an operator algebra, there exist a Hilbert space $\mathcal{H}$ and a completely isometric homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$. Observe that $\pi(f) = \langle(\pi(f_{ij}))v, w\rangle \in B(\mathcal{H})$. From this it follows that

$$\begin{aligned}
\|f\|_{\mathcal{A}} &= \|\pi(f)\| = \|\langle(\pi(f_{ij}))v, w\rangle\| \\
&= \sup\{|\langle[\langle(\pi(f_{ij}))v, w\rangle]\alpha, \beta\rangle| :\ \alpha, \beta \in \mathcal{H} \text{ such that } \|\alpha\| = 1,\ \|\beta\| = 1\} \\
&= \sup\{|\langle(\pi(f_{ij}))\alpha \otimes v, \beta \otimes w\rangle| :\ \alpha, \beta \in \mathcal{H} \text{ such that } \|\alpha\| = 1,\ \|\beta\| = 1\} \\
&\overset{*}{\leq} \|(\pi(f_{ij}))\|\|\alpha \otimes v\|\|\beta \otimes w\| = \|((f_{ij}))\|_{M_n(\mathcal{A})} \leq 1.
\end{aligned}$$

where (*) follows from the classical Cauchy-Schwarz inequality.

Hence, $\|F(p)\| = |f(p)| \leq d_{\mathcal{A}}(p, q)$, and thus by taking the the supremum over all F that satisfy the above properties we get that $d^n_{\mathcal{A}}(p, q) \leq d_{\mathcal{A}}(p, q)$, which is the required inequality.

The final conclusion follows by combining the above result with the obvious inequality that $d_{\mathcal{A}}(p, q) \leq d^n_{\mathcal{A}}(p, q)$ for every n. □

Traditionally, the concept of pseudodistance is closely tied with the two-point interpolation problem, for instance, see Theorem 5.3.1. The solution of the two-point interpolation problem for an algebra on some set X sometimes provides a neat way of calculating the pseudodistance on the set X induced by that algebra.

Interestingly but certainly not surprisingly, the pseudodistance also shows up in studying the two dimensional representations of an algebra of functions defined on some set X. In [56], a necessary and sufficient condition for a representation of an uniform algebra $\mathcal{A} \subseteq C(X)$ to be contractive was obtained in terms of the pseudodistance on the set X induced by the algebra $\mathcal{A}$. A closer look into their proof suggests that a similar result can be obtained for the two dimensional representations of an operator algebra of functions in terms of the corresponding pseudohyperbolic distance. It seems that the operator algebra of functions serves as an appropriate object for the generalization of this theorem.

We introduce some of the relevant notations that we require to state the theorem. Let $\mathcal{A}$ be an operator algebra of functions on some set X. Let $F = \{z_1, z_2\}$ be a two element subset of X and let $I_F := \{f \in \mathcal{A} : f(z_1) = 0 = f(z_2)\}$ be the ideal of functions in $\mathcal{A}$. Since $\mathcal{A}$ separates points of X, there exist functions $f_i \in \mathcal{A}$ such that $f_i(z_j) = \delta_i^j$ for every $i, j = 1, 2$. Then for every $(f_{ij}) \in M_n(\mathcal{A})$, we have that $(f_{ij} - f_{ij}(z_1)f_1 - f_{ij}(z_2)f_2) \in M_n(I_F)$.

This further implies that every $(f_{ij} + I_F) \in M_n(\mathcal{A}/I_F)$ can be written as $(f_{ij} + I_F) = (f_{ij}(z_1)f_1 + f_{ij}(z_2)f_2 + I_F)$. If $\pi : \mathcal{A}/I_F \to B(\mathcal{H})$ is a unital homomorphism then it is easy to see that $E_1 = \pi(f_1 + I_F)$ and $E_2 = \pi(f_2 + I_F)$ are idempotent operators that sum to the identity operator on $\mathcal{H}$. Further, we may decompose $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ is the range of the operator E_1 and write E_1 and E_2 as operator matrices with respect to this decomposition. We find that there exists a bounded operator $B : \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$\pi(E_1) = \begin{pmatrix} I_{\mathcal{H}_1} & B \\ 0 & 0 \end{pmatrix} \qquad \text{and } \pi(E_2) = \begin{pmatrix} 0 & -B \\ 0 & I_{\mathcal{H}_2} \end{pmatrix}. \tag{5.1}$$

We need a series of lemmas to prove the theorem, the one that follows is a standard result about J-contractions. We call an operator of the form $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(B(\mathcal{H}))$ J-contraction if $U^*JU \le J$, where $J = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix}$. For more details on this, we refer the reader to the notes by Paulsen [63].

Lemma 5.3.3. *Let $A, B, C, D, X \in B(\mathcal{H})$, be such that $CX + D$ is invertible and $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a J-contraction. If we define $\Psi_U(X) = (AX + B)(CX + D)^{-1}$, then $\|\Psi_U(X)\| < 1$ whenever $\|X\| < 1$.*

Proof. It follows from the fact $U^*JU \le J$ that $\begin{bmatrix} X^* & I \end{bmatrix} U^*JU \begin{bmatrix} X \\ 1 \end{bmatrix} \le \begin{bmatrix} X^* & I \end{bmatrix} J \begin{bmatrix} X \\ 1 \end{bmatrix}$. After performing block matrix multiplication, we get that

$$(AX + B)^*(AX + B) - (CX + D)^*(CX + D) \le X^*X - I_{\mathcal{H}}.$$

Since $\|X\| < 1$, we have that $X^*X - I_{\mathcal{H}} < 0$ and

$$(AX + B)^*(AX + B) < (CX + D)^*(CX + D).$$

Pre-multiplying by $(CX+D)^{*-1}$ and post-multiplying by $(CX+D)^{-1}$, the above equation turns into

$$(CX+D)^{*-1}(AX+B)^*(AX+B)(CX+D)^{-1} < 1 \Longrightarrow \|\Psi_U(X)\| < 1.$$

□

Lemma 5.3.4. *Let $\mathcal{A}$ be an operator algebra of functions on the set X. We assume that $\mathcal{A}$ is a Banach algebra. Let F and I_F be as in the discussion above the lemma. Then a unital homomorphism $\pi : \mathcal{A}/I_F \to B(\mathcal{H})$ is n-contractive if and only if $\|(\pi(f_{ij}))\| \leq \|(f_{ij}+I_F)\|$ for every $(f_{ij}) \in M_n(\mathcal{A})$ such that $(f_{ij}(z_1)) = 0$.*

Proof. Let π be a unital homomorphism such that $\|(\pi(f_{ij}))\| \leq \|(f_{ij}+I_F)\|$ for every $(f_{ij}) \in M_n(\mathcal{A})$ with $(f_{ij}(z_1)) = 0$. Let $G = (g_{ij}) \in M_n(\mathcal{A})$ be such that $\|G\| < 1$ and $G(z_1) \neq 0$. Note that $\|G(z)\| \leq \|G\| < 1$ for every $z \in X$. This together with the fact that $\mathcal{A}$ is a Banach algebra implies that $I - G(z_1)^*G(.)$ is invertible in $M_n(\mathcal{A})$. Thus, if we define $H(.) = (G(z_1) - G(.))(1 - G(z_1)^*G(.))^{-1}$, then $H \in M_n(\mathcal{A})$ and $H(z_1) = 0$. This further implies that $\|\pi^{(n)}(H)\| \leq \|H\|$.

Since $\mathcal{A}/I_F$ is an operator algebra, it can be represented on some Hilbert space $\mathcal{K}$ via a completely isometric map $\phi : \mathcal{A}/I_F \to B(\mathcal{K})$. We thus have that $\|\phi^{(n)}(G)\| = \|G\| < 1$. It is easy to see that $U = \begin{pmatrix} -I_{\mathcal{H}^n} & G(z_1)I_{\mathcal{H}^n} \\ -G(z_1)^*I_{\mathcal{H}^n} & I_{\mathcal{H}^n} \end{pmatrix}$ is a J-contraction. From the above lemma, we see that $\|\Psi_U(X)\| < 1$ for every bounded operator X with $\|X\| < 1$ and thus $\|\phi^{(n)}(H)\| = \|\Psi_U(\phi^{(n)}(G))\| < 1$. From this, it follows that $\|H\| < 1$ which implies that $\|\pi^{(n)}(H)\| < 1$.

Finally to prove $\|\pi^{(n)}(G)\| < 1$, note that $\pi^{(n)}(G) = \Psi_V(\pi^{(n)}(H))$, where

$V = \begin{pmatrix} I_{\mathcal{H}^n} & G(z_1)I_{\mathcal{H}^n} \\ G(z_1)^* I_{\mathcal{H}^n} & I_{\mathcal{H}^n} \end{pmatrix}$ is a J-contraction. This completes the proof of the result.

□

Theorem 5.3.5. *Let $\mathcal{A}$ be an operator algebra of functions on the set X such that $\mathcal{A}$ is also a Banach algebra. Let $F = \{z_1, z_2\}$ be a two element subset of X and I_F be the ideal of function in $\mathcal{A}$ that vanish on the set F. Then the unital homomorphism $\pi : H^\infty_{\mathcal{R}}(G)/I_F \to B(\mathcal{H})$ is completely contractive if and only if $\|B\| \leq (d_{\mathcal{A}}(z_1, z_2) - 1)^{-1/2}$ where B is the operator that appears in Equation (5.1). Moreover for the case $n = 1$, equality holds if and only if π is isometric.*

Proof. From the above lemma, it is enough to show that $\|(\pi(f_{ij}))\| \leq \|(f_{ij} + I_F)\|$ for every $(f_(ij)) \in M_n(\mathcal{A})$ with $(f_{ij}(z_1)) = 0$ and for every n if and only if $\|B\| \leq (d_{\mathcal{A}}(z_1, z_2) - 1)^{-1/2}$.

Let $(f_(ij)) \in M_n(\mathcal{A})$ such that $(f_{ij}(z_1)) = 0$ and $\|(f_{ij})\| \leq 1$. Then

$$\begin{aligned} \|(\pi(f_{ij} + I_F))\|^2 &= (\pi(f_{ij} + I_F))(\pi(f_{ij} + I_F))^* \\ &= [(f_{ij}(z_2)) \otimes \pi(E_1)][(f_{ij}(z_2)) \otimes \pi(E_1)]^* \\ &= \left\| \begin{bmatrix} 0 & 0 \\ 0 & (f_{ij}(z_2))(f_{ij}(z_2))^* \otimes (I_{\mathcal{H}_1} + BB^*) \end{bmatrix} \right\|. \end{aligned}$$

Thus,

$$\|(\pi(f_{ij} + I_F))\|^2 = \|(f_{ij}(z_2))\|^2 (1 + \|B\|^2). \tag{5.2}$$

First we assume that $\|B\| \leq ((d_{\mathcal{A}}(z_1, z_2))^{-2} - 1)^{1/2}$ then it immediately follows that

$$\|(\pi(f_{ij} + I_F))\|^2 = \|(f_{ij}(z_1))\|^2 (1 + \|B\|^2) \leq (d_{\mathcal{A}}(z_1, z_2))^2 (1 + \|B\|^2) \leq 1.$$

To prove the converse, we assume that π is completely contractive and take $f \in \mathcal{A}$ such that $f(z_1) = 0$ and $\|f\| \leq 1$. Then

$$\|f(z_1)\|^2(1 + \|B\|^2) = \|\pi(f + I_F)\|^2 \leq \|f + I_F\|^2 \leq 1.$$

By taking the supremum over all $f \in \mathcal{A}$ that satisfy $(f_{ij}(z_1)) = 0$ and $\|(f_{ij})\| \leq 1$, we get the required inequality. Note that it was enough to assume that π is contractive.

To prove the final assertion, we assume that $n = 1$. First, we claim that $|f(z_2)| = d_{\mathcal{A}}(z_1, z_2)\|f + I_F\|$ for every $f \in \mathcal{A}$ with $f(z_1) = 0$. Clearly, $|f(z_2)| \leq d_{\mathcal{A}}(z_1, z_2)\|f + I_F\|$. Let $g \in \mathcal{A}$ such that $g(z_1) = 0,\ g(z_2) \neq 0$ and $\|g\| < 1$, then $h(.) = -f(.) + \frac{f(z_2)}{g(z_2)}g(.) \in I_F$. Thus, $\|f + I_F\| \leq \|f + h\| = \left\|\frac{f(z_2)}{g(z_2)}g\right\| \leq \left|\frac{f(z_2)}{g(z_2)}\right|$. By taking the supremum over all such g's we get that $d_{\mathcal{A}}(z_1, z_2)\|f + I_F\| \leq |f(z_2)|$ which proves our claim.

Finally we get the result by using this claim together with the Equation 5.2,

$$\|\pi(f + I_F)\|^2 = \|f(z_2)\|^2(1 + \|B\|^2)\| = d^2_{\mathcal{A}}(z_1, z_2)\|f + I_F\|^2(1 + \|B\|^2)$$

and hence completes the proof of the result. □

Remark 5.3.6. *Note that the proof of the condition for an isometry in the above theorem requires that one finds a function h in $\mathcal{A}$ that satisfies $h(z_1) = 0$. Our method of constructing such a function is unsuitable for the generalization to matrix-valued functions.*

In [34], the authors explicitly constructed a completely isometric representation of two dimensional quotient uniform algebra into M_2 using the pseudo-metric on the set X induced by the algebra $\mathcal{A}$ which allows them to compute the C^*-envelope of this quotient algebra. A careful review of their proof brings out the interesting fact that it works perfectly well for any operator algebra of functions. We shall include it here for completeness, but before giving the proof of the theorem we shall prove the following lemma which will aid us in proving the theorem.

Lemma 5.3.7. *Let $W_i \in M_n$ for every $i = 1, 2$ and let $B : \mathcal{H}_2 \to \mathcal{H}_1$ be bounded operator from a Hilbert space $\mathcal{H}_2$ into a Hilbert space $\mathcal{H}_1$. Then*

$$\left\| \begin{bmatrix} W_1 \otimes I_{\mathcal{H}_1} & (W_1 - W_2) \otimes B \\ 0 & W_2 \otimes I_{\mathcal{H}_2} \end{bmatrix} \right\| = \left\| \begin{bmatrix} W_1 & (W_1 - W_2)\|B\| \\ 0 & W_2 \end{bmatrix} \right\|.$$

Proof. Denote $\begin{bmatrix} W_1 \otimes I_{\mathcal{H}_1} & (W_1 - W_2) \otimes B \\ 0 & W_2 \otimes I_{\mathcal{H}_2} \end{bmatrix}$ by T and $\begin{bmatrix} W_1 & (W_1 - W_2)\|B\| \\ 0 & W_2 \end{bmatrix}$ by S.

To prove $\|T\| = \|S\|$, we write a polar decomposition of B as $B = UP$ where U is a partial isometry and P is a positive operator such that $\|P\| = \|B\|$ and U^*U is a projection onto the closure of the range of P.

If we let $\mathcal{K}$ be any infinite dimensional Hilbert space, then by a reshuffling argument we get that

$$\left\| \begin{bmatrix} W_1 \otimes (I_{\mathcal{H}_1} \oplus I_{\mathcal{K}}) & (W_1 - W_2) \otimes (B \oplus 0_{\mathcal{K}}) \\ 0 \oplus 0_{\mathcal{K}} & W_2 \otimes (I_{\mathcal{H}_2} \oplus I_{\mathcal{K}}) \end{bmatrix} \right\| = \|T\|$$

where $0_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ is the zero map.

Note that $B \oplus 0 = (U \oplus 0_{\mathcal{K}})(P \oplus 0_{\mathcal{K}})$ and $U \oplus 0_{\mathcal{K}}$ is a partial isometry from the range of $P \oplus 0_{\mathcal{K}}$ onto the range of $B \oplus 0_{\mathcal{K}}$. Since $\mathcal{K}$ is an infinite dimensional Hilbert space, the complements of the range of $P \oplus 0$ and the range of $B \oplus 0_{\mathcal{K}}$ are infinite dimensional and consequently, there exists an unitary $V : (Ran(P \oplus 0_{\mathcal{K}}))^{\perp} \to (Ran(B \oplus 0_{\mathcal{K}}))^{\perp}$. Then $W = \begin{pmatrix} U \oplus 0_{\mathcal{K}} & 0 \\ 0 & V \end{pmatrix} : H_1 \oplus \mathcal{K} \to H_2 \oplus \mathcal{K}$ is an onto isometry.

It is now easy to see that

$$\left\|\begin{bmatrix} I\otimes W & 0 \\ 0 & I\otimes I \end{bmatrix}\begin{bmatrix} W_1\otimes(I_{\mathcal{H}_1}\oplus I_{\mathcal{K}}) & (W_1-W_2)\otimes(B\oplus 0_{\mathcal{K}}) \\ 0 & W_2\otimes(I_{\mathcal{H}_2}\oplus I_{\mathcal{K}}) \end{bmatrix}\begin{bmatrix} I\otimes W^* & 0 \\ 0 & I\otimes I \end{bmatrix}\right\|$$
$$= \left\|\begin{bmatrix} W_1\otimes(I_{\mathcal{H}_1}\oplus I_{\mathcal{K}}) & (W_1-W_2)\otimes(P\oplus 0_{\mathcal{K}}) \\ 0 & W_2\otimes(I_{\mathcal{H}_2}\oplus I_{\mathcal{K}}) \end{bmatrix}\right\|.$$

Let $C^*(P)$ be the C^*-algebra generated by $I_{\mathcal{H}_2}\oplus I_{\mathcal{K}}$ and $P\oplus 0_{\mathcal{K}}$ then it is commutative and is isometrically isomorphic to the space of continuous functions on $\sigma(P\oplus 0_{\mathcal{K}})$. From this it follows that $\begin{bmatrix} W_1\otimes(I_{\mathcal{H}_1}\oplus I_{\mathcal{K}}) & (W_1-W_2)\otimes(P\oplus 0_{\mathcal{K}}) \\ 0 & W_2\otimes(I_{\mathcal{H}_2}\oplus I_{\mathcal{K}}) \end{bmatrix} \in M_2(M_n(C^*(P)))$. Therefore, by functional calculus

$$\left\|\begin{bmatrix} W_1\otimes(I_{\mathcal{H}_1}\oplus I_{\mathcal{K}}) & (W_1-W_2)\otimes(P\oplus 0_{\mathcal{K}}) \\ 0 & W_2\otimes(I_{\mathcal{H}_2}\oplus I_{\mathcal{K}}) \end{bmatrix}\right\| = \sup_{t\in\sigma(P\oplus 0_{\mathcal{K}})}\left\|\begin{bmatrix} W_1 & (W_1-W_2)f(t) \\ 0 & W_2 \end{bmatrix}\right\|.$$

Finally, the result follows by recalling that $\|P\| = \|B\|$. □

Theorem 5.3.8. *Let X be any set and let $\mathcal{A}$ be an operator algebra of functions defined on X. If we let $F = \{z_1, z_2\}$ be a two element subset of X and set $b = (d_{\mathcal{A}}(z_1,z_2)^{-2}-1)^{1/2}$. Then the representation $\pi : \mathcal{A}/I_F \to M_2$ defined by*

$$\pi(f) = \begin{pmatrix} f(z_1) & (f(z_1)-f(z_2))b \\ 0 & f(z_2) \end{pmatrix}$$

is completely isometric.

Proof. Since $\mathcal{A}$ is an operator algebra, it is easy to see that $\mathcal{A}/I_F$ is an operator algebra and consequently there exist a Hilbert space $\mathcal{H}$ and a complete isometric representation $\phi : \mathcal{A}/I_F \to B(\mathcal{H})$. Recall, there exists a bounded operator B such that

$$\phi(E_1) = \begin{pmatrix} I_{\mathcal{H}_1} & B \\ 0 & 0 \end{pmatrix} \qquad \text{and } \phi(E_2) = \begin{pmatrix} 0 & -B \\ 0 & I_{\mathcal{H}_2} \end{pmatrix}$$

where E_1 and E_2 are bounded operators that span $\mathcal{A}/I_F$.

It is now immediate that $\|E_i\|^2 = \|\pi(E_i)\|^2 = 1 + \|B\|^2$ for every $i = 1, 2$. Recall that the pseudohyperbolic distance on $\mathcal{A}$ is given by

$$d_{\mathcal{A}}(z_1, z_2) = \sup\{|f(z_2)| : \|f\|_{\mathcal{A}} \leq 1,\ f(z_1) = 0\}.$$

Let $f \in \mathcal{A}$ such that $\|f\|_{\mathcal{A}} \leq 1,\ f(z_1) = 0$. Thus, the fact $\mathcal{A}/I_F = \text{span}\{E_1, E_2\}$ implies that $f + I_F = f(z_2)E_2$ and $\|f + I_F\| \leq 1$. From which it follows that $d_{\mathcal{A}}(z_1, z_2) \leq \|E_2\|^{-1}$. The other inequality immediately follows from the fact that $E_2 = f_2 + I_F$ and $f_2(z_i) = \delta_2^i$. Thus, we have that $d_{\mathcal{A}}(z_1, z_2) = \|E_2\|^{-1}$.

Since ϕ is a completely isometric representation on $\mathcal{A}$, we see that $\|(f_{ij} + I_F)\| = \|(\phi(f_{ij} + I_F))\|$ for every $(f_{ij}) \in M_n(\mathcal{A})$ and for all n. Note that $f_{ij} + I_F = f_{ij}(z_1)E_1 + f_{ij}(z_2)E_2$ for every $i,\ j$, which further implies that

$$\|(f_{ij}(z_1)) \otimes E_1 + (f_{ij}(z_2)) \otimes E_2\| = \left\| \begin{bmatrix} (f_{ij}(z_1)) \otimes I_{\mathcal{H}_1} & ((f_{ij}(z_1)) - (f_{ij}(z_2))) \otimes B \\ 0 & (f_{ij}(z_2)) \otimes I_{\mathcal{H}_2} \end{bmatrix} \right\|.$$

Finally by using Lemma 5.3.7, we see that

$$\|(f_{ij}(z_1)) \otimes E_1 + (f_{ij}(z_2)) \otimes E_2\| = \left\| \begin{bmatrix} (f_{ij}(z_1)) & ((f_{ij}(z_1)) - (f_{ij}(z_2)))\|B\| \\ 0 & (f_{ij}(z_2)) \end{bmatrix} \right\|.$$

Thus, if we define a homomorphism $\pi : \mathcal{A}/I_F \to M_2$ via the map

$$\pi(f + I_F) = \begin{bmatrix} f(z_1) & (f(z_1) - f(z_2))b \\ 0 & f(z_2) \end{bmatrix}$$

where $b = \|B\|$, then by the canonical reshuffling argument we get that

$$\|(\pi(f_{ij} + I_F))\| = \left\| \begin{bmatrix} (f_{ij}(z_1)) & ((f_{ij}(z_1)) - (f_{ij}(z_2)))\|B\| \\ 0 & (f_{ij}(z_2)) \end{bmatrix} \right\| = \|(f_{ij} + I_F)\|.$$

This completes the proof of the result. □

We believe that an interesting theory can be developed using these notions of pseudodistances but we do not intend to digress in that direction at this moment.

Instead, we would like to draw the reader's attention towards our particular example of an analytically presented domain. Fix $r < 1$, and $z_1,\ z_2 \in \mathbb{A}_r$, define

$$d_{H^\infty_{\mathcal{R}}(\mathbb{A}_r)}(z_1, z_2) := \sup\{|f(z_1)| : f \in H^\infty_{\mathcal{R}}(\mathbb{A}_r),\ \|f\|_{\mathcal{R}} < 1, f(z_2) = 0\}.$$

To avoid far too many subscripts, we denote $d_{H^\infty_{\mathcal{R}}(\mathbb{A}_r)}(z_1, z_2)$ by $d_r(z_1, z_2)$.

It is a standard exercise to show that $d_{H^\infty(\mathbb{D}^2)}(\zeta, \eta) = \max\{d_{H^\infty(\mathbb{D})}(\zeta_1, \eta_1), d_{H^\infty(\mathbb{D})}(\zeta_2, \eta_2)\}$, where $\zeta = (\zeta_1, \zeta_2)$ and $\eta = (\eta_1, \eta_2)$. In particular for every $z_1, z_2 \in \mathbb{A}_r$, we get that

$$d_{H^\infty(\mathbb{D}^2)}((z_1, r{z_1}^{-1}), (z_2, r{z_2}^{-1})) = \max\{d_{H^\infty(\mathbb{D})}(z_1, z_2), d_{H^\infty(\mathbb{D})}(rz_1^{-1}, rz_2^{-1})\}.$$

Before we prove a direct generalization of Schwarz-Pick lemma for $H^\infty_{\mathcal{R}}(\mathbb{A}_r)$, we have a quick observation in order.

Proposition 5.3.9. *If* $z_1, z_2 \in \mathbb{A}_r$ *then* $d_r(z_1, z_2) = \max\{d_{H^\infty(\mathbb{D})}(z_1, z_2), d_{H^\infty(\mathbb{D})}(rz_1^{-1}, rz_2^{-1})\}$, *and* $d_r^{cb}(z_1, z_2) = \max\{d^{cb}_{H^\infty(\mathbb{D})}(z_1, z_2), d^{cb}_{H^\infty(\mathbb{D})}(rz_1^{-1}, rz_2^{-1})\}$.

Proof. Let $f \in H^\infty(\mathbb{D}^2)$ such that $\|f\|_{\infty,\mathbb{D}^2} \leq 1$ and $f(z_2, rz_2^{-1}) = 0$. If we define $g : \mathbb{A}_r \to \mathbb{C}$ via $g(z) = f(z, rz^{-1})$, then $g \in H^\infty(\mathbb{A}_r)$ and $\|g\|_{\mathcal{R}} \leq \|f\|_{\infty,\mathbb{D}^2} \leq 1$. From this it follows that

$$d_{H^\infty(\mathbb{D}^2)}((z_1, rz_1^{-1}), (z_2, rz_2^{-1})) \leq d_r(z_1, z_2).$$

Conversely, let $f \in H^\infty_{\mathcal{R}}(\mathbb{A}_r)$ be such that $\|f\|_{\mathcal{R}} \leq 1$ and $f(z_2) = 0$. By using Theorem 5.2.2 there exists a function $g \in H^\infty_{\mathcal{R}}(\mathbb{D}^2)$ such that $\|g\|_{\infty,\mathbb{D}^2} \leq 1$ and $g(z, rz^{-1}) = f(z)$ for every $z \in \mathbb{A}_r$. Note that

$$|f(z_1)| = |g(z_1, rz_1^{-1})| \leq \sup\{|h(z_1, rz_1^{-1})| : h \in H^\infty(\mathbb{D}^2),\ \|h\|_{\infty,\mathbb{D}^2} \leq 1,\ h(z_2, r{z_2}^{-1}) = 0\}.$$

Finally by taking the supremum of the left hand side of the above inequality over all $f \in H^\infty_{\mathcal{R}}(\mathbb{A}_r)$ with $\|f\|_{\mathcal{R}} \le 1$ and $f(z_2) = 0$, we get that

$$d_r(z_1, z_2) \le d_{H^\infty(\mathbb{D}^2)}((z_1, rz_1^{-1}), (z_2, rz_2^{-1})).$$

This proves that

$$d_r(z_1, z_2) = \max\{d^{cb}_{H^\infty(\mathbb{D})}(z_1, z_2), d^{cb}_{H^\infty(\mathbb{D})}(rz_1^{-1}, rz_2^{-1})\}.$$

Note that the above argument hold for matrix-valued functions as well. □

As a direct consequence of the above proposition and general Schwarz-Pick lemma for the bidisk, we get

Theorem 5.3.10 (Schwarz-Pick lemma for the annulus)**.** *Given two points $\lambda_1,\ \lambda_2 \in \mathbb{A}_r$ and $w_1, w_2 \in \mathbb{C}$. Then there exists a function $f \in H^\infty_{\mathcal{R}}(\mathbb{A}_r)$ that maps λ_1 to w_1 and λ_2 to w_2 and $\|f\|_{\mathcal{R}} \le 1$ if and only if*

$$d_r(w_1, w_2) \le d_r(\lambda_1, \lambda_2).$$

Moreover, if f is biholomorphic then equality holds.

We now obtain another proof of the above theorem without appealing to the general Schwarz-Pick lemma for the bidisk by using simple but useful observations and the Schwarz-Pick lemma for the disk.

Lemma 5.3.11. *For any two points $\lambda_1,\ \lambda_2$ in $\mathbb{A}_r$, we have the following.*

(1) $|\lambda_1\lambda_2| \ge r$ *if and only if* $\left(\frac{1-r^2\overline{\lambda_i}^{-1}\lambda_j^{-1}}{1-\overline{\lambda_i}\lambda_j}\right) \ge 0.$

(2) $|\lambda_1\lambda_2| \le r$ *if and only if* $\left(\frac{1-\overline{\lambda_i}\lambda_j}{1-r^2\overline{\lambda_i}^{-1}\lambda_j^{-1}}\right) \ge 0.$

Proof. Note that both the matrices $A = \left(\frac{1-r^2\overline{\lambda_i}^{-1}\lambda_j^{-1}}{1-\overline{\lambda_i}\lambda_j}\right)$ and $B = \left(\frac{1-\overline{\lambda_i}\lambda_j}{1-r^2\overline{\lambda_i}^{-1}\lambda_j^{-1}}\right)$ are self adjoint and have positive diagonal entries for every $\lambda_1, \lambda_2 \in \mathbb{A}_r$. Thus, it is enough to show that $|\lambda_1\lambda_2| \geq r$ if and only if $det(A) \geq 0$ and $|\lambda_1\lambda_2| \leq r$ if and only if $det(B) \geq 0$. We see that

$$\begin{aligned} det(A) \geq 0 \iff & (|\lambda_1|^2 - r^2|)(|\lambda_2|^2 - r^2)(|1-\overline{\lambda_1}\lambda_2|^2) \geq |\overline{\lambda_1}\lambda_2 - r^2|^2(1-|\lambda_1|^2)(1-|\lambda_2|^2) \\ \iff & (1-r^2)\left[r^2(\overline{\lambda_1}\lambda_2 + \lambda_1\overline{\lambda_2}) + (|\lambda_1\lambda_2|^2 - r^2)(|\lambda_1|^2+|\lambda_2|^2) - |\lambda_1\lambda_2|^2(\overline{\lambda_1}\lambda_2 + \lambda_1\overline{\lambda_2})\right] \geq 0 \\ \iff & (1-r^2)|\lambda_1 - \lambda_2|^2(|\lambda_1\lambda_2|^2 - r^2) \geq 0 \iff |\lambda_1\lambda_2| \geq r. \end{aligned}$$

Note that the result follows from the observation that $det(B) \geq 0$ if and only if $det(A) \leq 0$. □

Recall that $d_{H^\infty(\mathbb{D})}(z_1, z_2) = \rho(z_1, z_2) = \left|\frac{z_1 - z_2}{1 - z_1\overline{z_2}}\right|$. A simple calculation yields the following result.

Lemma 5.3.12. *Given two points $\lambda_1,\ \lambda_2$ in $\mathbb{A}_r$, then*

$$d_r(\lambda_1, \lambda_2) = \begin{cases} \rho(r\lambda_1^{-1}, r{\lambda_2}^{-1}) & \text{if } |\lambda_1\lambda_2| \leq r \\ \rho(\lambda_1, \lambda_2) & \text{if } |\lambda_1\lambda_2| \geq r. \end{cases}$$

In particular, when $|\lambda_1\lambda_2| = r$ then $\rho(\lambda_1, \lambda_2) = \rho(r{\lambda_1}^{-1}, r{\lambda_2}^{-1})$.

Proof of Theorem 5.3.10:

We know by Theorem 5.2.7 that there exists a function $f \in H^\infty_{\mathcal{R}}(\mathbb{A}_r)$ such that $f(\lambda_i) = w_i$ for every i and $\|f\|_{\mathcal{R}} \leq 1$ if and only if there exist positive matrices (p_{ij}) and (q_{ij}) such that

$$1 - \overline{w_i}w_j = (1 - \overline{\lambda_i}\lambda_j)p_{ij} + (1 - r^2\overline{\lambda_i}^{-1}{\lambda_j}^{-1})q_{ij}.$$

If we now divide this equation by $(1-\overline{\lambda_i}\lambda_j)$ and assume that $|\lambda_1\lambda_2| \geq r$ then by Lemma 5.3.11 we get that $\left(\frac{1-\overline{w_i}w_j}{1-\overline{\lambda_i}\lambda_j}\right) \geq 0$. It follows from the classical Nevanlinna-Pick that $\left(\frac{1-\overline{w_i}w_j}{1-\overline{\lambda_i}\lambda_j}\right) \geq 0$

if and only if there exists $g \in H^{\infty}(\mathbb{D})$ such that $\|g\|_{\infty,\mathbb{D}} \leq 1$ and $g(\lambda_i) = w_i$ for every $i = 1, 2$. By the definition of norm, we have that $\|g\|_{\mathcal{R}} \leq \|g\|_{\infty,\mathbb{D}}$ which implies that $g \in H^{\infty}(\mathbb{A}_r)$ and $\|g\|_{\mathcal{R}} \leq 1$. This shows that for $\lambda_1, \lambda_2 \in \mathbb{A}_r$, that satisfies the inequality $|\lambda_1\lambda_2| \geq 0$, there exists a function $f \in H^{\infty}_{\mathcal{R}}(\mathbb{A}_r)$ such that $f(\lambda_i) = w_i$ for every i and $\|f\|_{\mathcal{R}} \leq 1$ if and only if $\left(\frac{1-\overline{w_i}^{-1}w_j^{-1}}{1-\overline{\lambda}_i\lambda_j}\right) \geq 0$. By using Theorem 5.3.1, we get that $\left(\frac{1-\overline{w_i}^{-1}w_j^{-1}}{1-\overline{\lambda}_i\lambda_j}\right) \geq 0 \iff \rho(w_1, w_2) \leq \rho(\lambda_1, \lambda_2)$.

Similarly, we can show that for $\lambda_1, \lambda_2 \in \mathbb{A}_r$, with $|\lambda_1\lambda_2| \geq 0$, there exists a function $f \in H^{\infty}_{\mathcal{R}}(\mathbb{A}_r)$ such that $f(\lambda_i) = w_i$ for every i and $\|f\|_{\mathcal{R}} \leq 1$ if and only if $\rho(w_1, w_2) \leq \rho(r{\lambda_1}^{-1}, r{\lambda_2}^{-1})$.

Combining the two together, we get that there exists a function $f \in H^{\infty}_{\mathcal{R}}(\mathbb{A}_r)$ such that $f(\lambda_i) = w_i$ for every i and $\|f\|_{\mathcal{R}} \leq 1$ if and only if

$$\rho(w_1, w_2) \leq \max\{\rho(\lambda_1, \lambda_2), \rho(r{\lambda_1}^{-1}, r{\lambda_2}^{-1})\} = d_r(\lambda_1, \lambda_2).$$

Before we close this section, we would like remark that our original motivation for introducing the concept of pseudohyperbolic metric on $H^{\infty}_{\mathcal{R}}(\mathbb{A}_r)$ was to develop a tool that allows us to tackle the long-standing problem of finding a satisfactory lower bound for the spectral constant for annulus. But we haven't been able to find any good result as yet. We will highlight the difficulties that arise in doing so in the last section of this chapter.

5.4 Spectral Constant

Spectral sets were introduced and studied by J. von Neumann [81] in 1951. The concept of spectral sets is partially motivated by von Neumann's inequality, which can be interpreted as saying that an operator T is a contraction if and only if the closed unit disk is a spectral set for T. Note that the closed annulus, $\overline{\mathbb{A}_r} = \{z : r \leq |z| \leq 1\}$ is the intersection of two closed disks: $\mathbb{D} = \{z : |z| \leq 1\}$ and $\mathbb{D}_r = \{z \in \mathbb{C} : \|z^{-1}\| \leq r^{-1}\}$ such that each closed disk is a spectral set. It is natural to wonder if the intersection of two spectral sets is a spectral set. Unfortunately, this is false, it was noticed by Shields [74] in 1974 and he proved that the annulus provides a counter-example. A very simple counter-example can be found in [59]. In the same article [74], Shields proved that the annulus is a K-spectral set where $K = 2 + (\frac{1+r}{1-r})^{1/2}$. The same proof of Shields shows that annulus is in fact K^{cb}-complete spectral set with $K^{cb} \leq K$. From this it follows that the two norms on $H^\infty(\mathcal{A}_r)$ are equivalent norms: $\|(f_{ij})\|_\infty \leq \|(f_{ij})\|_{\mathcal{R}} \leq K^{cb}\|(f_{ij})\|_\infty$ for every $(f_{ij}) \in M_n(H^\infty(\mathbb{A}_r))$.

In 1974, Shields raised the question of what is the smallest such constant and if this constant remains bounded as $r \to 1$? Very recently, this question has been answered positively that the optimal constant remains bounded in [16]. Their result states that

$$\frac{2}{1+r} \leq K_r \leq K_r^{cb} \leq 2 + \frac{\sqrt{r}+1}{\sqrt{\sqrt{r}+r+1}} \leq 2 + \frac{2}{\sqrt{3}}$$

where K_r^{cb} is the smallest complete spectral constant and K_r is the smallest spectral constant. In particular, we only know that the optimal constant independent of r lies between $\frac{4}{3}$ and $2 + \frac{2}{\sqrt{3}}$. Many researchers have worked on the problem of improving the bounds for spectral constant. Still, the problem of finding the optimal (complete) spectral constant remains open. We attempt to tackle this problem via two different approaches; a concrete approach and an abstract approach. A concrete approach uses pseudohyperbolic distance as its main tool and an abstract approach involves the use of hyperconvex sets which were introduced in [27]. Both of these approaches presents a viable method for finding an estimate for the lower bound of K_r and K_r^{cb}.

First, we present an approach that involves Nevanlinna-Pick interpolation and pseudo-hyperbolic distance. We know that

$$\|(f_{ij})\|_\infty \le \|(f_{ij})\|_{\mathcal{R}} \le K_r^{cb}\|(f_{ij})\|_\infty$$

for every $(f_{ij}) \in M_n(H^\infty(\mathbb{A}_r))$ and for every n. In particular for $n = 1,\ K_r^{cb}$ is replaced by K_r.

Recall the definition of GPHD for $H^\infty_{\mathcal{R}}(\mathbb{A}_r)$ and $H^\infty(\mathbb{A}_r)$ from Section 5.3,

$$d_r(z_1, z_2) = \sup\{|f(z_2)| : \|f\|_{\mathcal{R}} \le 1,\ f(z_1) = 0\}$$

and

$$d_{H^\infty(\mathbb{A}_r)}(z_1, z_2) = \sup\{|f(z_2)| : \|f\|_\infty \le 1,\ f(z_1) = 0\}.$$

It is now easy to see that for every $z_1, z_2 \in \mathbb{A}_r$, we have

$$d_r(z_1, z_2) \le d_{H^\infty(\mathbb{A}_r)}(z_1, z_2) \le K_r\ d_r(z_1, z_2).$$

Note that

$$K_r \ge \sup_{z_1 \ne z_2 \in \mathbb{A}_r} \left\{ \frac{d_{H^\infty(\mathbb{A}_r)}(z_1, z_2)}{d_r(z_1, z_2)} \right\}.$$

Since $K_r \le K_r^{cb}$, we have that

$$K_r^{cb} \ge K_r \ge \sup_{z_1 \ne z_2 \in \mathbb{A}_r} \left\{ \frac{d_{H^\infty(\mathbb{A}_r)}(z_1, z_2)}{d_r(z_1, z_2)} \right\}.$$

Thus, the problem of finding a lower bound of $K_r,\ K_r^{cb}$ is transformed into the problem of finding an estimate of the ratio of the "quantum" and the "classical" pseudohyperbolic distances on the annulus.

We can compute the classical pseudohyperbolic distance on the annulus by using Abrahamse's interpolation theorem [1]. According to which there exists $f \in H^\infty(\mathbb{A}_r)$ with

$f(z_1) = 0,\ \|f\|_\infty \leq 1$ and $f(z_2) = \lambda$ if and only if

$$\begin{bmatrix} K_t(z_1, z_1) & K_t(z_1, z_2) \\ K^t(z_2, z_1) & (1 - |\lambda|^2)K_t(z_2, z_2) \end{bmatrix} \geq 0 \text{ for all } t \in \mathbb{R}$$

where $K_t(z_1, z_2) = \sum_{n=-\infty}^{\infty} \dfrac{(z_1\overline{z_2})^n}{1 + r^{2n+1-2t}}$.

Note that the the latter condition holds if and only if the determinant of the matrix is non-negative, that is,

$$|\lambda|^2 \leq 1 - \frac{|K_t(z_1, z_2)|^2}{K_t(z_1, z_1)K_t(z_2, z_2)} \text{ for all } t \in \mathbb{R}.$$

This further implies that

$$d_r(z_1, z_2)^2 = \inf_{t\in\mathbb{R}} \left\{1 - \frac{|K_t(z_1, z_2)|^2}{K_t(z_1, z_1)K_t(z_2, z_2)}\right\}.$$

We can obtain a more useful distance formula by using a result by Fedorov and Vinnikov [40] in which they showed that once the points $z_1, \cdots, z_n \in \mathbb{A}_r$ are fixed there exist two points $t_0,\ t_1$ such that the positivity of the matrices $[(1 - z_i\overline{z_j})K_{t_0}(z_i, z_j)]$ and $[(1 - z_i\overline{z_j})K_{t_1}(z_i, z_j)]$ guarantees the existence of a scalar-valued solution. They were also able to show that the parameters t_0 ad t_1 depend on the points $z_1, \cdots, z_n \in \mathbb{A}_r$ and are given by $t_\beta = \sum_{i=1}^{n} \frac{log|z_1|}{2log(r)} - \frac{1}{2}\beta,\ \beta = 0, 1$. In addition to [40], we refer the reader to the paper by McCullough [53] for further details on this result.

Thus, we see that

$$d_r(z_1, z_2)^2 = \min_{\beta=0,1} \left\{1 - \frac{|K_{t_\beta}(z_1, z_2)|^2}{K_{t_\beta}(z_1, z_1)K_{t_\beta}(z_2, z_2)}\right\}. \tag{5.3}$$

It might be of value to use this formula to estimate the lower bound of K_r, but the tools required to compute it seems beyond us at this time.

A classical way to compute the distance formula for the annulus which uses Green's function first appeared in [31]. In [31], the authors computed the distance formula for the annulus $\mathbb{A}(1/R, R)$ which has inner radius $1/R$ and outer radius R and for the points $z_1, z_2 \in \mathbb{A}(1/R, R)$ where $1/R < z_1 < R$. Later in [73], the authors computed the distance formula $d_{H^\infty(\mathbb{A}_r)}(\rho, z_2)$ when $r < \rho < 1$ for the annulus $\mathbb{A}_r$, by using the invariance of this pseudohyperbolic distance under biholomorphic maps. Thus, from this result, we know that the minimum in the Equation 5.3 is equal to this formula when $z_1 = \rho$. Because of the complicated form of $d_{H^\infty(\mathbb{A}_r)}(\rho, z_2)$, we only state it for a particular set of points in the annulus.

For instance, take $z_1 = \sqrt{r},\ z_2 = \sqrt{r}e^{i\theta}$ for some $\theta \in (0, 2\pi)$ then from the result that appeared in [31], we have that

$$d_{H^\infty(\mathbb{A}_r)}(\sqrt{r}, \sqrt{r}e^{i\theta}) = 2\sqrt{r}\sqrt{2 - 2cos\theta}\ \prod_{n=1}^{\infty} f_n g_n(\theta)$$

where $f_n = \left(\dfrac{1 + r^{2n}}{1 + r^{2n-1}}\right)^2$ and $g_n(\theta) = \dfrac{1 - 2r^{2n}cos\theta + r^{4n}}{1 - 2r^{2n-1}cos\theta + r^{4n-2}}$.

We now use this formula of $d_{H^\infty(\mathbb{A}_r)}$ to estimate the lower bound of K_r.

Also, we know by Lemma 5.3.12 that

$$d_r(z_1, z_2) = \begin{cases} \rho(rz_1^{-1}, r{z_2}^{-1}) & \text{if } |z_1 z_2| \le r \\ \rho(z_1, z_2) & \text{if } |z_1 z_2| \ge r. \end{cases}$$

Thus, for $z_1 = \sqrt{r},\ z_2 = \sqrt{r}e^{i\theta}$, we get that

$$d_r(z_1, z_2) = \rho(\sqrt{r}, \sqrt{r}e^{i\theta}) = \sqrt{\frac{r(2 - 2cos\theta)}{1 + r^2 - 2rcos\theta}}.$$

Finally, we get that

$$K_r \ge \sup_{z_1 \ne z_2 \in \mathbb{A}_r} \left\{ \frac{d_{H^\infty(\mathbb{A}_r)}(z_1, z_2)}{d_r(z_1, z_2)} \right\} \ge 2\sqrt{1 + r^2 - 2rcos\theta}\ \prod_{n=1}^{\infty} f_n g_n(\theta) \text{ for every } \theta \in (0, 2\pi),$$

where f_n and $g_n(\theta)$ are as above.

In particular, if we take $\theta = \pi$ and $\theta = \pi/2$ then after simplifying we get

$$K_r \geq 2(1+r)\ \left(\prod_{n=1}^{\infty} \frac{1+r^{2n}}{1+r^{2n-1}}\right)^4$$

and

$$K_r \geq 2\sqrt{1+r^2}\ \prod_{n=1}^{\infty} \frac{(1+r^{4n})(1+r^{2n})^2}{(1+r^{4n-2})(1+r^{2n-1})^2}.$$

Calculating the lower bound of K_r requires us to estimate the lower bound of the infinite product in the above equation. A simple calculation shows that the above bound is greater than the existing lower bound, $\frac{2}{1+r}$. Thus, as of this writing, the estimate $\frac{2}{1+r}$ is the best known estimate. We are currently working on this problem. We believe that the above approach together with the classical work done on infinite products and infinite series can be worthy in finding a better estimate than $\frac{2}{1+r}$.

We now present our second approach, albeit an abstract approach, to find an estimate of the smallest spectral constant for the annulus. This involves hyperconvex sets which were introduced in [27].

Let $A \subseteq C(X)$ be a uniform algebra, fix a finite subset $F = \{x_1, x_2, \cdots, x_n\} \subseteq X$ and let I_F denote the ideal of functions in $\mathcal{A}$ that vanish on the set F, $I_F = \{f \in A : f(z_i) = 0\ \forall\ 1 \leq i \leq n\}$. Cole, Lewis, and Wermer define the set

$$\mathfrak{D}(A; x_1, \cdots, x_n) = \{(w_1, w_2, \cdots, w_n) : \|f + I_F\| \leq 1,\ f(z_i) = w_i \text{ for every } 1 \leq i \leq n\}.$$

Such sets were also called interpolation body. It was shown in [75] that this serves as the natural coordinatization of the closed unit ball of the operator algebra A/I_F. In [60], Paulsen introduced the concept of matricial hyperconvex sets which serves as the natural coordinatization of the closed unit ball of the operator algebra $M_n(A/I_F)$. We now extend these definitions for operator algebras of functions. Given an operator algebra of functions

$\mathcal{A}$ on some set X, a finite subset $F = \{z_1, z_2 \cdots, z_n\} \subseteq X$ and let I_F denote the ideal of functions in $\mathcal{A}$ that vanish on the set $F = \{f \in \mathcal{A} : f(z_i) = 0 \ \forall \ 1 \leq i \leq n\}$. Then the hyperconvex set for $\mathcal{A}/I_F$ is defined as

$$\mathfrak{D}(\mathcal{A}; x_1, \cdots, x_n) = \{(w_1, w_2, \cdots, w_n) : \|f + I_F\|_{\mathcal{A}} \leq 1,\ f(z_i) = w_i \text{ for every } 1 \leq i \leq n\}$$

and the matricial hyperconvex set as

$$\mathfrak{D}^k(\mathcal{A}; x_1, \cdots, x_n) = \{(W_1, W_2, \cdots, W_n) : \|(f_{ij}+I_F)\|_{M_k(\mathcal{A})} \leq 1,\ f(z_i) = W_i \text{ for every } 1 \leq i \leq n\}.$$

Obviously, $\mathfrak{D}^1(\mathcal{A}; x_1, \cdots, x_n) = \mathfrak{D}(\mathcal{A}; x_1, \cdots, x_n)$.

The next proposition highlights the relation between hyperconvex sets and the quotient operator algebras.

Proposition 5.4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two operator algebras of functions on the same set X and let $F = \{x_1, x_2, \cdots, x_n\}$ be a finite subset of X. Then $\mathfrak{D}^k(\mathcal{A}; x_1, \cdots, x_n) \subseteq C\mathfrak{D}^k(\mathcal{B}; x_1, \cdots, x_n)$ implies that $\|(f_{ij} + I_F(\mathcal{B}))\|_{M_k(\mathcal{B})} \leq C\|(f_{ij} + I_F(\mathcal{A}))\|_{M_k(\mathcal{A})}$, where $I_F(\mathcal{A}) = \{f \in \mathcal{A} : f(x_i) = 0 \ \forall \ 1 \leq i \leq n\}$ and $I_F(\mathcal{B}) = \{f \in \mathcal{B} : f(x_i) = 0 \ \forall \ 1 \leq i \leq n\}$.*

Proof. Let $(W_1, W_2, \cdots, W_n) \in \mathfrak{D}^k(\mathcal{A}; x_1, \cdots, x_n)$, then there exists an $(f_{ij}) \in M_k(\mathcal{A})$ such that $\|(f_{ij} + I_F(\mathcal{A}))\|_{M_k(\mathcal{A})} \leq 1$ and $f_{ij}(x_l) = W_l$ for every $1 \leq l \leq n$. It follows from the hypotheses that $\|(f_{ij} + I_F(\mathcal{B}))\|_{M_k(\mathcal{B})} \leq C$, and thus we have that $(\frac{W_1}{C}, \frac{W_2}{C}, \cdots, \frac{W_n}{C}) \in \mathfrak{D}^k(\mathcal{B}; x_1, \cdots, x_n)$. This implies that $\mathfrak{D}^k(\mathcal{A}; x_1, \cdots, x_n) \subseteq C\mathfrak{D}^k(\mathcal{B}; x_1, \cdots, x_n)$. □

To prove our main result in this context, we turn towards our particular setting. Let G be analytically presented domain. Recall,

$$H^\infty_{\mathcal{R}}(G) = \{f \in H^\infty(G) : \|f\|_{\mathcal{R}} \leq 1\},$$

where $\|f\| = \sup\{\|f(T)\| : T \in \mathcal{Q}(G)\}$ and $\mathcal{Q}(G)$ is the quantized version of G. As a shorthand, we write $\mathfrak{D}_\infty(G; z_1, \ldots, z_n)$ for $\mathfrak{D}_\infty(H^\infty(G); z_1, \ldots, z_n)$ and $\mathfrak{D}_\mathcal{R}(G; z_1, \ldots, z_n)$ for $\mathfrak{D}_\mathcal{R}(H^\infty(G); z_1, \ldots, z_n)$. We define

$$k_n(G; z_1, \ldots, z_n) = \inf\{C : \mathfrak{D}_\infty(G; z_1, \ldots, z_n) \subseteq C\mathfrak{D}_\mathcal{R}(G; z_1, \ldots, z_n)\}$$

and

$$k_n(G) = \sup\{k_n(G; z_1, \ldots, z_n) : \{z_1, \ldots, z_n\} \subseteq G\}.$$

For the matrix-analogue, we define

$$k_n^l(G; z_1, \ldots, z_n) = \inf\{C : \mathfrak{D}_\infty^l(G; z_1, \ldots, z_n) \subseteq C\mathfrak{D}_\mathcal{R}^l(G; z_1, \ldots, z_n)\}$$

and

$$k_n^l(G) = \sup\{k_n^l(G; z_1, \ldots, z_n) : \{z_1, \ldots, z_n\} \subseteq G\}.$$

Let G be an analytically presented domain. Then in the view of the fact that $\sigma(T) \subseteq G$ for every $T \in \mathcal{Q}(G)$, we redefine the notion of K-spectral set for analytically presented domains.

Definition 5.4.2. *An analytically presented domain G is called a joint complete K-spectral set for $\mathcal{Q}(G)$ if $\|(f_{ij})\|_\mathcal{R} \leq K\|(f_{ij})\|_\infty$ for every $(f_{ij}) \in M_n(H^\infty(G))$. We call an analytically presented domain G a joint K-spectral set for $\mathcal{Q}(G)$ if $\|f\|_\mathcal{R} \leq K\|f\|_\infty$ for every $f \in H^\infty(G)$.*

Clearly, the annulus is an example of a joint complete K-spectral set. It is known that the intersection of two closed disks of the Riemann sphere is a K-spectral set. If we choose an appropriate analytic presentation of a domain then the intersection of two closed disk turn out to be a joint complete K-spectral set. For instance, the domain in Example 3.5.6

is a joint complete K-spectral set. Even for this example, not much is known in regards to the optimal spectral constant.

If G is a joint complete K-spectral set for which the smallest such constant K exists, then we denote the smallest K for which G is a joint complete K-spectral set by $K_o^{cb}(G)$ and we use $K_o(G)$ for the optimal constant for which G is a joint K_o-spectral set. In the set notation,

$$K_o^{cb}(G) = \inf\{C : \|(f_{ij})\| \leq C\|(f_{ij})\|_\infty \text{ for all } (f_{ij}) \in M_n(H^\infty(G)) \text{ and for all } n\}.$$

and

$$K_o(G) = \inf\{C : \|f\| \leq C\|f\|_\infty \text{ for all } f \in H^\infty(G)\}.$$

If $G = \mathbb{A}_r$, then $K_o^{cb}(\mathbb{A}_r) = K_r^{cb}$ and $K_o(\mathbb{A}_r) = K_r$.

Theorem 5.4.3. *Let G be a joint complete K-spectral set for which the smallest K exists. Then the sequence $\{k_n^l(G)\}_n$ defined above is an increasing sequence of real numbers with respect to n and $\sup_l\{\lim_n k_n^l(G)\} = K_o^{cb}(G)$. In particular $\lim_n k_n(G) = K_o(G)$.*

Proof. First, we show that for each l

$$k_n^l(G; z_1, z_2, \dots, z_n) \leq k_{n+1}^l(G; z_1, z_2, \dots, z_n, z_{n+1})$$

for every $z_1, \dots,\ z_n,\ z_{n+1} \in G$.

Let $(W_1, \dots, W_n) \in \mathfrak{D}_\infty(z_1, \dots, z_n)$, then there exists an $F = (f_{ij}) \in M_l(H^\infty(\mathbb{A}_r))$ such that $\|F + I_{\{z_1,\dots,\ z_n\}}\|_\infty \leq 1$ and $F(z_i) = W_i\ \forall\ i = 1, \dots, n$.

Let C' be such that $\mathfrak{D}_\infty(G; z_1, \dots, z_n, z_{n+1}) \subseteq C'\mathfrak{D}_{\mathcal{R}}(G; z_1, \dots, z_n, z_{n+1})$. Note that

$$(W_1, \dots, W_n, (f_{ij}(z_{n+1}))) \in \mathfrak{D}_\infty(G; z_1, \dots, z_n, z_{n+1})$$

which further implies that there exists a function $G = (g_{ij}) \in M_l(H^\infty_{\mathcal{R}}(\mathbb{A}_r))$ such that $\|(g_{ij} + I_{\{z_1,\ldots,z_n,z_{n+1}\}})\| \leq C'$, $G(z_{n+1}) = F(z_{n+1})$ and $G(z_i) = W_i$ for every $i = 1, \ldots, n$.

It is easy to see that the ideal of functions obey the containment rule, $I_{\{z_1,\ldots,z_n\}} \subseteq I_{\{z_1,\ldots,z_n,z_{n+1}\}}$. This implies that $\|(g_{ij} + I_{\{z_1,\ldots,z_n\}})\|_{\mathcal{R}} \leq \|(g_{ij} + I_{\{z_1,\ldots,z_n,z_{n+1}\}})\|_{\mathcal{R}}$. Since $\|(g_{ij} + I_{\{z_1,\ldots,z_n,z_{n+1}\}})\|_{\mathcal{R}} \leq C'$, we get that $\|(g_{ij} + I_{\{z_1,\ldots,z_n\}})\|_{\mathcal{R}} \leq C'$.

It follows from $f_{ij} - g_{ij} \in I_{\{z_1,\ldots,z_n\}}$ for every i, j that $\|(f_{ij} + I_{\{z_1,\ldots,z_n\}})\|_{\mathcal{R}} = \|(g_{ij} + I_{\{z_1,\ldots,z_n\}})\|_{\mathcal{R}} \leq C'$. Thus, we find that $\|(f_{ij} + I_{\{z_1,\ldots,z_n\}})\|_{\mathcal{R}} \leq C'$ which further implies that $\mathfrak{D}^l_\infty(G; z_1, \cdots, z_n) \subseteq C'\mathfrak{D}^l_{\mathcal{R}}(G; z_1, \cdots, z_n)$ for every l. From this, we get $k^l_n(G; z_1, z_2, \ldots, z_n) \leq C'$ for every C' that satisfies

$$\mathfrak{D}^l_\infty(G; z_1, \cdots, z_n, z_{n+1}) \subseteq C'\mathfrak{D}^l_{\mathcal{R}}(G; z_1, \cdots, z_n, z_{n+1}).$$

By taking the infimum over all such C', we obtain

$$k^l_n(G; z_1, \ldots, z_n) \leq k^l_{n+1}(G; z_1, \ldots, z_n, z_{n+1})$$

for every $\{z_1, \ldots, z_n\} \subseteq G$ and for every l. This proves that $\{k^l_n(G)\}_n$ is an increasing sequence of positive real numbers for every l.

Lastly, we need to prove $\sup_l\{\lim_n k^l_n(G)\} = K^{cb}_o(G)$. Obviously, we have that

$$\mathfrak{D}^l_\infty(G; z_1, \cdots, z_n) \subseteq k^l_n\mathfrak{D}^l_{\mathcal{R}}(G; z_1, \cdots, z_n).$$

It is easy to see that each $k^l_n(G)$ is bounded above by $K^{cb}_o(G)$ which is a finite number by the assumption. Therefore, $\lim_n k^l_n(G) = \sup_n k^l_n(G) \leq K^{cb}_o(G)$ for every l.

By Proposition 5.4.1, we see that

$$\|(f_{ij} + I_F)\|_{\mathcal{R}} \leq \sup_n k^l_n(G)\|(f_{ij} + I_F)\|_{\mathcal{R}}$$

for every $(f_{ij}) \in M_l(H^\infty(\mathbb{G}))$ and for every finite subset $F \subseteq \mathbb{G}$. We know that $H^\infty_{\mathcal{R}}(G)$ is a local operator algebra of functions which follows from Theorem 3.4.11. Thus, we get that

$$\|(f_{ij})\|_{\mathcal{R}} \leq \sup_n k_n^l \|(f_{ij})\|_{\mathcal{R}}$$

for every $(f_{ij}) \in M_l(H^\infty(\mathbb{G}))$ and for every l. Since $K_o^{cb}(G)$ is the optimal joint complete K-spectral constant, we get that $K_o^{cb}(G) \leq \sup_l\{\sup_n k_n^l(G)\}$. This proves that $\sup_l\{\lim_n k_n^l(G)\} = K_o^{cb}(G)$ and hence completes the proof of the result. □

The above theorem indicates a strong connection between the problem of estimating spectral constants and interpolation problems. Certainly, the above approach is not the most effective way to compute an estimate on the upper bound of the spectral constant but it might be of some value when computing the estimate on the lower bound of the spectral constant.

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